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IMAGE PROCESSING BY MEANS OF ŁUKASIEWICZ ALGEBRA WITH SQUARE ROOT

Annotation of the Ph.D. Thesis

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Abstract

My doctoral thesis is oriented to digital 2D image de-noising as part of digital 2D image processing. A good mathematical model for image processing background represents Łukasiewicz algebra with square root. After its definition, terms of fuzzy logic function and its sensitivity were introduced. The sensitivity propositions and their proofs follow.

Basic terms of digital image processing and description of several image processing techniques are subjects of second part of my doctoral thesis. A quality of image processing can be measured using Minkowski distance of processed and ideal images. A multicriterial approach can be applied if several quality measures are used.

A sample of traditional image filters, which are not realizable in Łukasiewicz algebra with square root, is included in the third part first. Then a set of filters, which are realizable in Łukasiewicz algebra with square root, is discussed. Finally, a list of fuzzy logic function filters is introduced.

The image de-noising can be performed by hierarchical structures for data processing. This structure is called fuzzy logic network if it contains fuzzy logic functions only in hidden nodes. Five fuzzy logic networks including their sensitivity are introduced. Their outputs can be better than results of individual fuzzy filters.

Both individual and hierarchical fuzzy filter properties are subject of experimental part. The set of seven artificial images is used in tests. Four quality measures are used for fuzzy filter comparing. Then the multicriterial approach is applied and the set of eight prime filters is selected. This set is used as first hidden layer of fuzzy logic networks. Their structures are optimized in the second step. The results of three types of fuzzy logic networks are better than result of the best individual fuzzy filter. Finally, real biomedical images are processed via selected individual fuzzy filters and all fuzzy logic networks.

All algorithms were realized in MATLAB environment.

Abstrakt

Má disertační práce je zaměřena na potlačování šumu ve 2D obrazech v kontextu digitálního zpracování 2D obrazů. Vhodným matematickým modelem pro zpracování obrazů je Łukasiewiczova algebra s odmocninou. Po její definici jsou uvedeny pojmy fuzzy logická funkce a její citlivost. Následují tvrzení o citlivosti a jejich důkazy.

Základní pojmy z oblasti digitálního zpracování obrazu a popis několika technik pro zpracování obrazu jsou předmětem druhé části mé disertační práce. Kvalitu zpracování obrazu lze měřit pomocí Minkowského vzdálenosti mezi zpracovaným a ideálním obrazem. V případě použití více různých měřítek kvality současně je možno aplikovat vícekriteriální přístup.

Ve třetí části je nejprve uvedeno několik tradičních obrazových filtrů, které nejsou realizovatelné v Łukasiewiczově algebře s odmocninou. Poté je diskutována množina filtrů, jež v Łukasiewiczově algebře s odmocninou realizovat lze. Nakonec je uveden seznam fuzzy filtrů.

Potlačování šumu v obrazech je možno provádět pomocí hierarchických struktur pro zpracování dat. Jestliže ve skrytých uzlech jsou obsaženy pouze fuzzy logické funkce, nazveme tuto strukturu fuzzy logickou sítí. Je uvedeno pět typů fuzzy logických sítí včetně jejich citlivosti. Jejich výstupy mohou být lepšími než výsledky jednotlivých fuzzy filtrů.

Předmětem experimentální části jsou vlastnosti individuálních i hierarchických filtrů. Při testování je použita množina sedmi umělých obrazů. Pro porovnání fuzzy filtrů je užito čtyř měřítek kvality. Pomocí vícekriteriálního přístupu je vybráno osm nejlepších filtrů, které jsou použity v první skryté vrstvě fuzzy logických sítí. Ve druhém kroku je optimalizována jejich struktura. Výsledky tří typů fuzzy logických sítí jsou lepší než výsledek nejlepšího individuálního fuzzy filtru. Nakonec jsou individuální i hierarchické fuzzy filtry použity pro odstranění šumu v reálných biomedicínských obrazech.

Všechny algoritmy byly realizovány v prostředí MATLABu.

Keywords

Digital Image Processing, Digital Image Enhancement, Digital Image De-Noising, 2D Image, Łukasiewicz Algebra with Square Root, Fuzzy Logic Function, Constrained Sensitivity, Linear and Nonlinear Filter, Fuzzy Logic Network, Biomedical Image, Magnetic Resonance Imaging

List of symbols

$\mathbf{N}$	set of natural numbers
$\mathbf{N}_0$	set of natural numbers including zero
$\mathbf{Z}$	set of integer numbers
$\mathbf{R}$	set of real numbers
$\mathbf{L}$	[0,1] interval
$\mathcal{X}$	pixel
x	pixel value/vector
r	row number
c	column number
$\mathcal{P} = (\mathcal{X}_{i,j})_{\substack{i=1,\dots,r \\ j=1,\dots,c}}$	image
$X = (x_{i,j})_{\substack{i=1,\dots,r \\ j=1,\dots,c}}$	image represented as matrix of pixel values/vectors

List of abbreviations

AIA	absolute value of distance from ideal alternative
BES	best easy systematic estimation
BL-algebra	basic logic algebra
CRNN	constrained referential neural network
CT	computed tomography
DFT	discrete Fourier transform
DWNN	dyadic weights neural network
FIR	finite impulse response
FLE	fuzzy logic expression
FLF	fuzzy logic function
IDFT	inverse discrete Fourier transform
IIR	infinite impulse response
ŁA_{sqrt}	Łukasiewicz algebra with square root
MAE	mean of absolute error
MAX	maximum of absolute error
MED	median of absolute error
MFLFN	modular FLF network
MMFN	min-max fuzzy network
MPFN	modus ponens fuzzy network
MRI	magnetic resonance imaging
MSE	mean square of error
MV-algebra	many-valued logic algebra
PET	positron emission tomography
SNR	signal/noise ratio
SPECT	single-photon emission computed tomography

Introduction

A 2D image processing is an important instrument for many fields of human activities. Biomedical, environmental, meteorological and technological structures are typical data sources. Their analysis has a wide application. The 2D image processing includes various operations, for example image reconstruction, filtering, sharpening and edge detection. Several techniques can be chosen for this purpose. One of them is a fuzzy logic.

The first aim of my doctoral thesis is to demonstrate that basic logic algebra—Łukasiewicz algebra with square root is useful mathematical background for image processing. The second aim is to develop hierarchical structures as fuzzy logic networks and then compare them with individual fuzzy filters for noise suppressing.

A first part is oriented to the algebraic model for data processing. Łukasiewicz algebra with square root was chosen from basic logic algebras as a tool for realization of fuzzy logic expressions and fuzzy logic functions. Their properties were studied in the metric space on $[0,1]^n$. The main result of this section is a sensitivity definition and computing of upper bounds of basic function sensitivities.

A second part describes basic terms of 2D image processing. Beginning with pixel, 2D image, neighborhood and mask, the role of list and weighted lists in image processing is defined. After the definition of noise, ideal, real and artificial images, the image de-noising is defined and the role of fuzzy logic functions in local image processing is discussed. The quality of image processing is also a subject of this part. The Pareto optimality and AIA technique are introduced in case of optimum de-noising of a set of images.

In a third part, traditional image filters are divided into two groups—filters, which are not realizable in ŁA_{sqrt} and filters, which are fuzzy logic functions. Last theoretical part introduces five approaches to hierarchical fuzzy logic function for image processing.

The experimental part is specialized to biomedical image de-noising. The MRI, PET, CT and SPECT techniques are described first. Then the artificial and ideal MRI and SPECT images are prepared for a testing of fuzzy logic function filters. Various fuzzy logic functions, quality measures and masks are used individually first. Finally, the five types of new hierarchical networks from previous part are used and optimized on artificial and ideal images. The resulting filters are used for de-noising of real MRI images.

The principles of data processing are realized in MATLAB environment and the source code examples are also included.

1 Ł$\mathbf{A}_{\text{sqrt}}$ as Algebraic Model

Łukasiewicz algebra is an MV-algebra operating on $[0,1]$ interval using conjunction, disjunction, multiplication and residuum as basic logic operators. Having no more than these operators, it is impossible to construct low sensitivity systems and compromise data processing. That is why the Łukasiewicz algebra was enriched by a square root function. This function is defined as extended inversion of Łukasiewicz square function.

DEFINITION: Let $\mathbf{L} = [0,1]$. Let $a, b \in \mathbf{L}$. Let

$$\begin{aligned} a \wedge b &= \min(a,b), \\ a \vee b &= \max(a,b), \\ a \otimes b &= \max(a+b-1,0), \\ a \rightarrow b &= \min(1-a+b,1), \\ \text{sqrt}(a) &= (1+a)/2. \end{aligned}$$

Then the ***Łukasiewicz algebra with square root*** (ŁA_{sqrt}) is defined as

$$\text{Ł}A_{sqrt} = \langle \mathbf{L}, \wedge, \vee, \otimes, \rightarrow, \text{sqrt}, 0, 1 \rangle.$$

It is useful to define some derived operators for the simplification of expression and function description. Some of them have a linguistic meaning, for example negation, equivalence, 'very' (the square) and 'roughly' (the square root).

DEFINITION: Let $n \in \mathbf{N}_0$. Let $a, b \in \mathbf{L}$. Then ***derived operators*** are defined as

$$\begin{aligned} \neg a &= a \rightarrow 0, \\ a \leftrightarrow b &= (a \rightarrow b) \wedge (b \rightarrow a), \\ a \circ b &= \neg(a \leftrightarrow b), \\ a \oplus b &= \neg(\neg a \otimes \neg b), \\ a \ominus b &= a \otimes \neg b, \\ n \odot a &= \bigoplus_{k=1}^{n} a = \underbrace{a \oplus a \oplus \cdots \oplus a}_{n},\ \ 0 \odot a = 0, \\ a^n &= \bigotimes_{k=1}^{n} a = \underbrace{a \otimes a \otimes \cdots \otimes a}_{n},\ \ a^0 = 1. \end{aligned}$$

All the operators in ŁA_{sqrt} can be represented by functions of one or two variables.

DEFINITION: Let $x, y \in \mathbf{L},\ n \in \mathbf{N}_0$. Then

$$\begin{aligned} \text{f}_1(x) &= \neg x \\ \text{f}_2(x) &= \text{sqrt}(x) \\ \text{f}_3(x,y) &= x \wedge y \\ \text{f}_4(x,y) &= x \vee y \\ \text{f}_5(x,y) &= x \otimes y \\ \text{f}_6(x,y) &= x \rightarrow y \end{aligned}$$

$$\begin{aligned} \text{f}_7(x,y) &= x \leftrightarrow y \\ \text{f}_8(x,y) &= x \circ y \\ \text{f}_9(x,y) &= x \oplus y \\ \text{f}_{10}(x,y) &= x \ominus y \\ \text{f}_{11}(x,n) &= n \odot x \\ \text{f}_{12}(x,n) &= x^n \end{aligned}$$

are defined as ***basic functions*** in ŁA_{sqrt}.

DEFINITION: ***Fuzzy logic expression*** (FLE) is defined by the rules:

(i) Any free variable $x \in \mathbf{L}$ is FLE.

(ii) Any constant $a \in \mathbf{L}$ is FLE.

(iii) $\mathrm{f}_i(\mathrm{FLE})$ is FLE for $i = 1, 2$,

(iv) $\mathrm{f}_j(\mathrm{FLE}, \mathrm{FLE})$ is FLE for $j = 3, \ldots, 10$,

(v) $\mathrm{f}_k(\mathrm{FLE}, n)$ is FLE for $k = 11, 12$ and $n \in \mathbf{N}_0$

where f_m $(m = 1, \ldots, 12)$ are the basic functions in ŁA$_{\mathrm{sqrt}}$.

DEFINITION: Let $n \in \mathbf{N}$, $\boldsymbol{x} \in \mathbf{L}^n$ and $\varphi : \mathbf{L}^n \to \mathbf{L}$. If $\varphi(\boldsymbol{x})$ is FLE then φ is called a ***fuzzy logic function*** (FLF) in ŁA$_{\mathrm{sqrt}}$.

Thus, fuzzy logic function in ŁA$_{\mathrm{sqrt}}$ is composed from constants and free variables from $\mathbf{L}$ and finite number of basic ŁA$_{\mathrm{sqrt}}$ operators and functions.

2 Properties of FLFs in Metric Space

Minkowski Metric Space

DEFINITION: Let $n \in \mathbf{N}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Let $p \in [1, \infty) \cup \{\infty\}$ be an exponent. Then a function

$$\mathrm{d}_p(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{1/p}, & \text{for } p \in [1, \infty), \\ \lim_{p \to \infty} \mathrm{d}_p(\boldsymbol{x}, \boldsymbol{y}) = \max_{k=1,\ldots,n} |x_k - y_k|, & \text{for } p = \infty. \end{cases}$$

is called ***Minkowski metric***.

The pair $\langle \mathbf{R}^n, \mathrm{d}_p \rangle$ is called ***Minkowski metric space***.

Unit Vector and Normalization

DEFINITION: Let $n \in \mathbf{N}$, $p \in [1; \infty) \cup \{\infty\}$ be parameters of Minkowski metric space $\langle \mathbf{R}^n, \mathrm{d}_p \rangle$. Let $\mathbf{0} \in \mathbf{R}^n$ be zero vector. Then $\boldsymbol{u} \in \mathbf{R}^n$ is called ***unit vector*** just when $\mathrm{d}_p(\boldsymbol{u}, \mathbf{0}) = 1$.

DEFINITION: Let $\langle \mathbf{R}^n, \mathrm{d}_p \rangle$ be the Minkowski metric space. Let

$$U_p = \{\boldsymbol{x} \in \mathbf{R}^n |\ \mathrm{d}_p(\boldsymbol{x}, \mathbf{0}) = 1\}$$

be a set of all unit vectors. The application of any function $\mathrm{t}^* : \mathbf{R}^n \to U_p$ to vector $\boldsymbol{x} \in \mathbf{R}^n$ is called a ***normalization*** in Minkowski metric space just when

$$\mathrm{d}_p(\boldsymbol{x}, \mathrm{t}^*(\boldsymbol{x})) = \mathrm{d}_p^*$$

where

$$\mathrm{d}_p^* = \min_{\boldsymbol{u} \in U_p} \mathrm{d}_p(\boldsymbol{x}, \boldsymbol{u}).$$

Applying Minkowski metrics to $\mathbf{L}^n$ space, we obtain the pair $\langle \mathbf{L}^n, \mathrm{d}_p \rangle$. This space will be used for the analysis of properties of any FLF.

THEOREM: Let $n \in \mathbf{N}$. Then d_∞ distance function is FLF on $\mathbf{L}^n$.

THEOREM: Let $\boldsymbol{x} \in \mathbf{L}^n$. Then a function $\mathrm{u} : \mathbf{L}^n \to \mathbf{L}$ such as $\boldsymbol{u} = \mathrm{u}(\boldsymbol{x}) = \boldsymbol{x} + 1 - \mathrm{d}_\infty(\boldsymbol{x}, \mathbf{0})$ is a FLF normalization in $\langle \mathbf{L}^n, \mathrm{d}_\infty \rangle$.

Lipschitz continuity

DEFINITION: Let $n \in \mathbf{N}$ and $\varphi : \mathbf{L}^n \to \mathbf{L}$. The φ is called ***Lipschitz continuous function on*** $\mathbf{L}$ if

$$(\exists \lambda \in \mathbf{R}_0^+)(\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{L}^n)\ \mathrm{d}(\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y})) \leq \lambda \cdot \mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) \tag{1}$$

where d is any metric.

It is possible to use the city-block distance d_1 as the metric and then write the formula (1) as

$$(\exists \lambda \in \mathbf{R}_0^+)(\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{L}^n)\ \varphi(\boldsymbol{x}) \circ \varphi(\boldsymbol{y}) \leq \lambda \cdot \sum_{k=1}^{n} |x_k - y_k|.$$

The Lipschitz continuity implies a continuity of function.

THEOREM: Any FLF is the Lipschitz continuous function.

Sensitivity

DEFINITION: Let $\varphi : \mathbf{L}^n \to \mathbf{L}$ be FLF. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{L}^n$. Then

$$\lambda = \max_{\boldsymbol{x} \neq \boldsymbol{y}} \frac{\varphi(\boldsymbol{x}) \circ \varphi(\boldsymbol{y})}{\sum_{k=1}^{n} |x_k - y_k|}$$

is called a ***sensitivity***.

THEOREM: Let $a \in \mathbf{L}$ be a constant, $\boldsymbol{x} \in \mathbf{L}^n$ and φ is FLF. Then the sensitivity of $\varphi(\boldsymbol{x}) = a$ is $\lambda_a = 0$ while sensitivity of $\varphi_j(\boldsymbol{x}) = x_j$ is $\lambda_{x_j} \leq 1$.

THEOREM: Let $\mathrm{p}, \mathrm{q} : \mathbf{L}^n \to \mathbf{L}$ be a Lipschitz continuous functions with sensitivities λ_p and λ_q. Then

$$\begin{aligned}
\lambda_{\neg \mathrm{p}} &= \lambda_\mathrm{p} \\
\lambda_{\mathrm{sqrt(p)}} &= \lambda_\mathrm{p}/2 \\
\lambda_{\mathrm{p} \wedge \mathrm{q}} &\leq \max(\lambda_\mathrm{p}, \lambda_\mathrm{q}) \\
\lambda_{\mathrm{p} \vee \mathrm{q}} &\leq \max(\lambda_\mathrm{p}, \lambda_\mathrm{q}) \\
\lambda_{\mathrm{p} \otimes \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q} \\
\lambda_{\mathrm{p} \to \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q}
\end{aligned}
\qquad
\begin{aligned}
\lambda_{\mathrm{p} \leftrightarrow \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q} \\
\lambda_{\mathrm{p} \circ \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q} \\
\lambda_{\mathrm{p} \oplus \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q} \\
\lambda_{\mathrm{p} \ominus \mathrm{q}} &\leq \lambda_\mathrm{p} + \lambda_\mathrm{q} \\
\lambda_{n \odot \mathrm{p}} &\leq n \cdot \lambda_\mathrm{p} \\
\lambda_{\mathrm{p}^n} &\leq n \cdot \lambda_\mathrm{p}
\end{aligned}$$

3 Processing of 2D Gray Image

Pixel

DEFINITION: The smallest discrete rectangular element of picture is called a ***pixel***.

The pixel has two properties: a size (this property is dependent on resolution of the apparatus which was used for picture digitizing) and a color (a property visible on a screen). The size is common property of all pixels in given picture, while the color is typically different for each pixel.

DEFINITION: Let $\mathcal{X}$ be a given pixel, $n \in \mathbf{N}$ and $M \subset \mathbf{R}$. Then a vector $x \in M^n$, which describes the color property of given pixel, is called a ***pixel vector*** (related to $\mathcal{X}$).

In case of $n = 1$, the pixel vector is called a ***pixel value*** or pixel intensity.

Each pixel is represented by its pixel vector in digital 2D image processing.

2D Image

DEFINITION: Let $r, c \in \mathbf{N}$. Then the matrix of size (r, c), which consists of $r \cdot c$ pixels, is called a 2D digital ***image***.

An image is a discrete 2D signal. From a mathematical point of view, the image is described as matrix with r rows and c columns where each element $x_{i,j} \in M^n$ ($n \in \mathbf{N}$, $M \subset \mathbf{R}$, $1 \leq i \leq r$, $1 \leq j \leq c$) is the pixel vector. Therefore, the image can be stored in a computer memory, and manipulated by a processor.

There are several kinds of images (depending on length of pixel vector n and range of its values M), for example:

- RGB color image: $n = 3$ and $M = [0, 1]$,
- gray image (graylevel or grayscale image): $n = 1$ and $M = [0, 1]$,
- binary image: $n = 1$, $M = \{0, 1\}$.

Grayscale images are very known as a subject of image processing. In following text, the all terms and definitions will be concerned with gray images.

Pixel Neighborhood

DEFINITION: Let $R \in \mathbf{N}_0$, $\mathcal{P}$ be any 2D gray image of size (r, c), $i, j \in \mathbf{N}$, $1 \leq i \leq r$ and $1 \leq j \leq c$. Let $\mathcal{X}_{i,j}$ be a given pixel from $\mathcal{P}$. Then a subset

$$\mathcal{N}_R(\mathcal{X}_{i,j}) = \{\mathcal{X}_{u,v} \in \mathcal{P}; |u - i| \leq R \wedge |v - j| \leq R\}$$

is called a ***pixel neighborhood*** and $\mathcal{X}_{i,j}$ is called a ***central pixel***.

From a graphical point of view, the pixel neighborhood is a small square matrix of odd size $(2R + 1)^2$ where the original pixel is just in the matrix center (Fig. 1).

The pixel neighborhood is also called a *window*. The R is called a *radius of pixel neighborhood*. There is $R \leq 3$ in many applications.

In the case of $R > 0$, there is necessary to extend the original image by left and right columns and top and bottom rows to prevent the degeneration of the neighborhood. The extension should be realized by zero padding, constant replication, periodic replication or mirror replication (Fig. 2).

The pixel neighborhood plays an important role in the pixel-by-pixel enhancement of the whole image.

$\mathcal{X}_{i-1,j-1}$	$\mathcal{X}_{i-1,j}$	$\mathcal{X}_{i-1,j+1}$
$\mathcal{X}_{i,j-1}$	$\mathcal{X}_{i,j}$	$\mathcal{X}_{i,j+1}$
$\mathcal{X}_{i+1,j-1}$	$\mathcal{X}_{i+1,j}$	$\mathcal{X}_{i+1,j+1}$

Figure 1: Pixel neighborhood for $R = 1$ with a central pixel $\mathcal{X}_{i,j}$

(a) (b) (c) (d)

Figure 2: Image extension for $R = 5$: (a) zero padding, (b) constant replication, (c) periodic replication, (d) mirror replication

Mask

DEFINITION: Let $R \in \mathbf{N}_0$ and $r = 2R + 1$. Then the square matrix $\mathbf{M} = (m_{i,j})$ of size (r, r), which consists of non-negative integer numbers, is called a ***mask***.

Let $k \in \mathbf{N}_0$, $k = \sum_{i=1}^{r} \sum_{j=1}^{r} m_{i,j}$. Then k is called a ***mask capacity***.

The mask $\mathbf{M} = (m_{i,j})$ of type (r, r) can be represented as vector of weights $\boldsymbol{w}$ of size $n = r^2$ using formula $w_k = m_{i,j}$ where $k = (i - 1) \cdot r + j$ $(i, j = 1, \ldots, r)$.

The R is called a *mask radius*. In case of local processing of gray image, the R determines a radius of pixel neighborhood. Then a weighted list of k pixel values where k is mask capacity can be formed from a pixel neighborhood using each mask element as frequency of appropriate pixel in the list.

DEFINITION: Let $R \in \mathbf{N}_0$, $r = 2R + 1$ and $\mathbf{M} = (m_{i,j})$ be a mask of size (r, r).

(a) The $\mathbf{M}$ is called a ***box mask***, iff $m_{i,j} = 1$ for $i, j = 1, 2, \ldots, r$.

(b) The $\mathbf{M}$ is called a ***binomial mask***, iff $m_{i,j} = \binom{r-1}{i-1} \cdot \binom{r-1}{j-1}$ for $i, j = 1, 2, \ldots, r$.

List and Weighted Lists

DEFINITION: Let $n \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{L}^n$. Then n-tuple $L = (x_1, \ldots, x_n)$ is called a ***list***. The list can be also denoted as $L = \overset{n}{\underset{k=1}{\Diamond}} x_k$.

The pixel vectors in given pixel neighborhood can be denoted as the list of pixel vectors.

DEFINITION: Let $n \in \mathbf{N}$, $\boldsymbol{x} \in \mathbf{L}^n$ and $\boldsymbol{w} \in \mathbf{N}_0^n$. Then a ***weighted list*** is defined as $L = \overset{n}{\underset{k=1}{\Diamond}} w_k x_k$ where $w_k x_k = (\underbrace{x_k, x_k, \ldots, x_k}_{w_k})$. The $\boldsymbol{w}$ is called a ***vector of weights***.

The weighted list can be formed from given pixel neighborhood using any mask with the same radius.

Local Processing

DEFINITION: Let $\mathcal{P}$ be any 2D gray image of size (r, c), $\mathcal{X}_{i,j}$ be a given pixel from $\mathcal{P}$, $R \in \mathbf{N}_0$ be a radius of $\mathcal{X}_{i,j}$ neighborhood, $\mathbf{M} = (m_{i,j})$ be any mask with radius R and mask capacity $k > 1$. Then a ***local processing*** is equivalent to a mapping $\mathrm{f} : \mathbf{L}^k \to \mathbf{L}$ where $x^*_{i,j} = \mathrm{f}(\boldsymbol{w})$ is new intensity (value) of central pixel $\mathcal{X}_{i,j}$ and $\boldsymbol{w} = (w_1, \ldots, w_k)$ is weighted list, which was formed from the $\mathcal{X}_{i,j}$ neighborhood using the mask $\mathbf{M}$.

Function f is called a ***local processing function***.

In case of any gray image enhancement, the data from each pixel neighborhood are typically proceed to obtain a new pixel intensities $x^*_{i,j}$. A pixel-by-pixel processing leads to the resulting enhanced gray image.

2D Image Processing

The image processing is a discipline of computer vision dealing with transformations of image data into image data. The design of an image transformation follows special aims, e.g. image segmentation, image enhancement, or image restoration.

The *image enhancement* is the processing of images to improve their appearance to human viewers or to enhance other image processing modules' performance. The objective of image enhancement is dependent on the application context and criteria for enhancement is often subjective or too complex to be easily converted to useful objective measures. Enhancement tasks are typical for the analysis of microscope images in medicine or biology, for material inspection, or for remote sensing.

Image enhancement is used in all image processing applications, it includes contrast manipulation, *image de-noising*, image smoothing, edge sharpening, pseudocoloring and so on.

Image De-Noising

DEFINITION: ***Image noise*** is defined as any unwanted disturbance in image data.

The noise is produced from the signal digitization, data recording and data transmission. Image noise may also result in "holes" in the image data.

DEFINITION: The model of 2D signal is called an ***ideal image***.

Any image obtained by measurement is called a ***real image***.

Any ideal image modified by any noise signal is called an ***artificial image***.

The ideal image can be approximated by any real image with minimum noise level.

DEFINITION: The special case of image enhancement, aim of which is the noise reduction and image structure saving, is called the ***image de-noising***.

The noise reduction and image structure saving are contradictory aims.

DEFINITION: Any neighborhood operator (image transformation), which decrease the noise level, is called a ***filter***.

Many filters can be described as local processing with given mask and given local processing function, which can be FLF in several cases.

Local FLF Processing for Image De-Noising

Individual FLF Processing

DEFINITION: Any local processing, where local processing function is FLF, is called a ***FLF processing***.

Thus, FLF processing is any FLF application to the weighted list formed on given image using given mask. A scheme of FLF processing is depicted in Fig. 3.

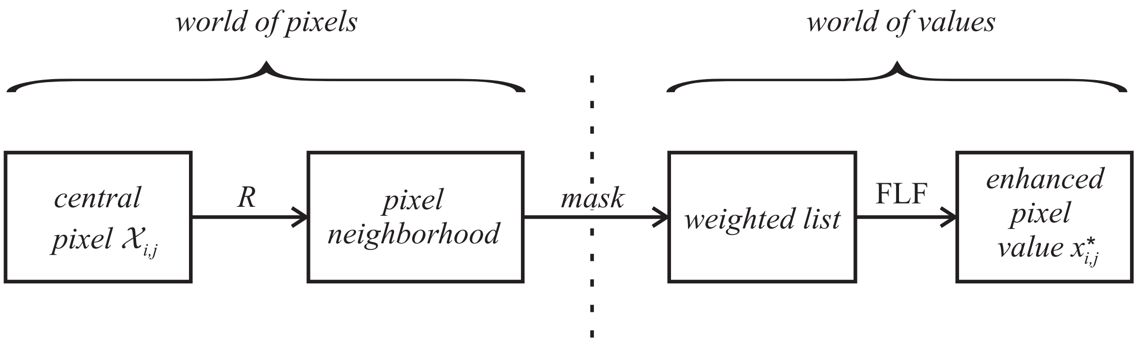

Figure 3: Local FLF processing scheme

Hierarchical FLF Processing

We can perform the local FLF processing as a hierarchical structure of data processing. Resulting structures are called FLF networks. They consist of independent layers with interconnections. The signals from pixel neighborhood come into the first input layer. The enhanced pixel is produced by output node in output layer. It is necessary to use one hidden layer at least for advanced pixel processing.

Realizing FLF networks, ŁA_{sqrt} can be used for node processing.

4 Quality of Image Processing

Image Similarity

DEFINITION: Let X and Y are 2D images of the same type where X is the ideal image and Y is noised one (it is identified as subject of image processing). Then $D = Y - X$ is called an ***image difference***.

The aim of image processing is to construct a new image Y^* such that $Y^* - X$ is approximately equal to zero.

DEFINITION: Let $r, c \in \mathbf{N}$. Let X, Y be the images of size (r, c) and $p \in [1, \infty) \subset \mathbf{R}$. Then

$$\mathrm{d}_p(X, Y) = \left(\sum_{i=1}^{r} \sum_{j=1}^{c} |y_{ij} - x_{ij}|^p \right)^{\frac{1}{p}}$$

is called a ***Minkowski distance of images*** X, Y.

The standard measures of image processing quality are derived from Minkowski distance of images: mean of absolute error (MAE; $p = 1$), mean square of error (MSE; $p = 2$) and maximum of absolute error (MAX; $p \to \infty$).

The *signal to noise ratio* (SNR) criterion corresponds with digital filtering standard. It is defined by a formula

$$\mathrm{SNR} = 10 \cdot \log \frac{\mathrm{Var}(X)}{\mathrm{Var}(Y - X)}$$

or by an alternative formula

$$\mathrm{SNR}^* = 10 \cdot \log \frac{\mathrm{Var}(X)}{\mathbf{E}(Y - X)^2}$$

where $\mathrm{Var}(A) = \mathbf{E}\{(A - \mathbf{E}A)^2\}$, where $\mathbf{E}$ is a symbol of average value.

Fifth criterion is a *median of absolute error* (MED).

Multicriterial Approach

Supposing an existence of single image in both noised and ideal form, the noised form of image can be passed through any filter and the quality of de-noising can be measured by various criteria. Then the quality of given filter is a vector of selected criteria values. The results of various filters can be collected to quality matrix. There is a possibility to find optimum filter using multicriteria decision making technique. Pareto optimality and AIA technique are introduced in my doctoral thesis.

5 Traditional Non-FLF Filters for Image De-Noising

The image processing is a traditional application of various mathematical methods. Several filters that are not realizable in ŁA_{sqrt} in general:

- mean filter (box filter),
- alpha trimmed mean filter,
- integer weighted average filter,
- Gaussian filter,
- Wiener filter,
- IIR filter.

6 Traditional FLF Filters for Image De-Noising

Statistics Functions as FLFs

The intensities of pixels from neighborhood are collected in various lists—basic terms (list, weighted list, sorted list) are introduced in my thesis.

THEOREM: Let $k, n \in \mathbf{N}$, $k \leq n$ and $\boldsymbol{x} \in \mathbf{L}^n$. Let $\psi_k : \mathbf{L}^n \to \mathbf{L}$ such that $\psi_k(\boldsymbol{x}) = x_{(k)}$ where $x_{(k)}$ is the k-th value from sorted list $\text{SORT}(L) = (x_{(1)}, \ldots, x_{(n)})$. Then ψ_k is FLF.

This theorem is useful for extended data processing (construction of the filters based on sorted value lists). The alternative data processing techniques prefer large lists of compromise values. The Walsh list is one of them.

DEFINITION: Let $n \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{L}^n$. Then the ***Walsh list*** is defined as

$$\mathrm{W}(\boldsymbol{x}) = \mathop{\Diamond}_{i \leq j} \left(\frac{x_i + x_j}{2} \right).$$

DEFINITION: Let $L = \mathop{\Diamond}\limits_{k=1}^{n} x_k$ be a list, $x_k \in \mathbf{L}$. Then the ***median*** of list L is defined as

$$\mathrm{M}(\boldsymbol{x}) = \operatorname*{median}_{i=1,\ldots,n} x_i = \frac{x_{(\lfloor \frac{n+1}{2} \rfloor)} + x_{(\lceil \frac{n+1}{2} \rceil)}}{2},$$

where $\lfloor a \rfloor = \max\{n \in \mathbf{N}; n \leq a\}$, $\lceil a \rceil = \min\{n \in \mathbf{N}; n \geq a\}$ and $x_{(k)}$ is the k–th value of L.

The local processing, which use any mask $\mathbf{M}$ with radius R and median as local processing function, is called a ***median filter***.

DEFINITION: Let $L = \mathop{\Diamond}\limits_{k=1}^{n} w_k x_k$ be a weighted list, $x_k \in \mathbf{L}$ and $w_k \in \mathbf{N}_0$. Then the ***weighted median*** is defined as median of weighted list.

Let $L = \mathop{\lozenge}\limits_{k=1}^{n} x_k$ be a list, $x_k \in \mathbf{L}$. Then the ***quasi median*** of list L is defined as

$$\mathrm{Q}_k(\boldsymbol{x}) = \frac{x_{(\lfloor \frac{n+1}{2} \rfloor - k)} + x_{(\lceil \frac{n+1}{2} \rceil + k)}}{2},$$

where $k \in \mathbf{N}$ and $k < \lfloor \frac{n+1}{2} \rfloor$.

The median of Walsh list is called ***Hodges-Lehmann median***:

$$\mathrm{HL}(\boldsymbol{x}) = \mathrm{median}\left\{\frac{x_i + x_j}{2}, 1 \le i \le j < n\right\}.$$

DEFINITION: Let $L = \mathop{\lozenge}\limits_{k=1}^{n} x_k$ be a list, $x_k \in \mathbf{L}$. Then the ***best easy systematic estimation*** (BES) is defined as

$$\mathrm{BES}(\boldsymbol{x}) = \frac{1}{4} \cdot \left(x_{(\lceil \frac{n}{4} \rceil)} + x_{(\lfloor \frac{n+1}{2} \rfloor)} + x_{(\lceil \frac{n+1}{2} \rceil)} + x_{(\lfloor \frac{3n+4}{4} \rfloor)} \right).$$

The BES estimation of Walsh list is called ***Walsh-BES estimation***.

THEOREM: The medians, quasi medians, Hodges-Lehmann median, BES and Walsh-BES estimation are FLFs.

Morphological Operators as FLFs

The field of mathematical morphology contributes a wide range of operators to image processing, all based around a few simple mathematical concepts from set theory. The operators are particularly useful for the analysis of binary or gray images and common usages include edge detection, noise removal, image enhancement and image segmentation.

Erosion is one of the two basic operators in the area of mathematical morphology, the other being dilation. It is typically applied to binary images, but there are versions that work on grayscale images:

DEFINITION: Let $\mathcal{P}$ be an image, $\mathbf{M}$ be a mask with radius R and $k \in \mathbf{N}$ be a capacity of $\mathbf{M}$. Let $\mathrm{ERO} : \mathbf{L}^k \to \mathbf{L}$,

$$\mathrm{ERO}(\boldsymbol{x}) = \bigwedge_{i=1}^{k} x_i.$$

Then a local processing using the mask $\mathbf{M}$ and local processing function ERO is called an ***erosion***.

The mask is called a *structuring element* in a field of mathematical morphology. A typical value of radius is $R = 1$. A larger mask produces a more extreme erosion effect. The weights in mask greater than one have no effect in any morphological operators.

DEFINITION: Let $\mathcal{P}$ be an image, $\mathbf{M}$ be a mask with radius R and k be a capacity of $\mathbf{M}$. Let $\mathrm{DIL} : \mathbf{L}^k \to \mathbf{L}$,

$$\mathrm{DIL}(\boldsymbol{x}) = \bigvee_{i=1}^{k} x_i.$$

Then a local processing using the mask $\mathbf{M}$ and local processing function DIL is called a ***dilation***.

Opening is defined as an erosion followed by a dilation using the same structuring element (mask) for both operations. Opening results in removal of narrow peaks. The initial erosion removes the small details and darkens the image. The following dilation increases the brightness but does not reintroduce the details removed by erosion.

Closing is defined simply as a dilation followed by an erosion using the same structuring element (mask) for both operations. Closing is used to remove dark details from an image. The initial dilation removes dark details and makes the image brighter. The erosion that follows darkens the image but does not reintroduce the details removed by dilation.

Dyadic FIR Filter

LEMMA: The division into any power of two is FLF.

THEOREM: Let $n \in \mathbf{N}$. Let $\boldsymbol{x} \in \mathbf{L}^n$. Let $N, m_k \in \mathbf{N}_0$ for $k = 1, \ldots, n$. Let $w_k = m_k/2^N$ be dyadic weights for $k = 1, \ldots, n$ and $\sum_{k=1}^{n} w_k \leq 1$. Then any function

$$\mathrm{f}(\boldsymbol{x}) = \sum_{k=1}^{n} w_k \cdot x_k$$

is a FLF.

DEFINITION: Let $n \in \mathbf{N}$, $N \in \mathbf{N}_0$, $\boldsymbol{x} \in \mathbf{L}^n$, $\boldsymbol{m} \in \mathbf{N}_0^n$, $\boldsymbol{w} = \boldsymbol{m}/2^N$ and $\sum\limits_{k=1}^{n} w_k \leq 1$. Then the FLF

$$\mathrm{f}(\boldsymbol{x}) = \sum_{k=1}^{n} m_k/2^N \cdot x_k$$

is called ***dyadic FIR filter***.

THEOREM: Any dyadic FIR filter satisfies the Lipschitz condition with the sensitivity

$$\lambda \leq \max_{k=1,\ldots,n} w_k.$$

Applying dyadic FIR filter to ordered Walsh list is an inspirational way how to construct new generation of FLF filters.

7 FLF Networks for Image De-Noising

The image processing using the local FLF one can be perceived as a hierarchical process without loops. Its representation by oriented acyclic graphs is recommended. A resulting FLF structure is called *fuzzy logic function network*. It consists of independent layers with interconnections. The signals from pixel neighborhood come into the first input layer. The enhanced pixel signal is produced by output node in output layer. It is necessary to use one hidden layer at least for advanced signal processing.

Modus Ponens Fuzzy Network

The modus ponens rule as the basic principle of logic motivates a four-layered fuzzy network architecture. A resulting structure is called a ***Modus Ponens Fuzzy Network*** (MPFN). The first MPFN layer contains n input nodes. The second layer consist of H hidden FLF nodes which make the FLF preprocessing with constrained sensitivity. The third layer realizes the modus ponens law using $2m$ hidden nodes and learnable weights. The fourth layer with m output nodes produces the compromise solution of given task. The MPFN structure is described in the Fig. 4.

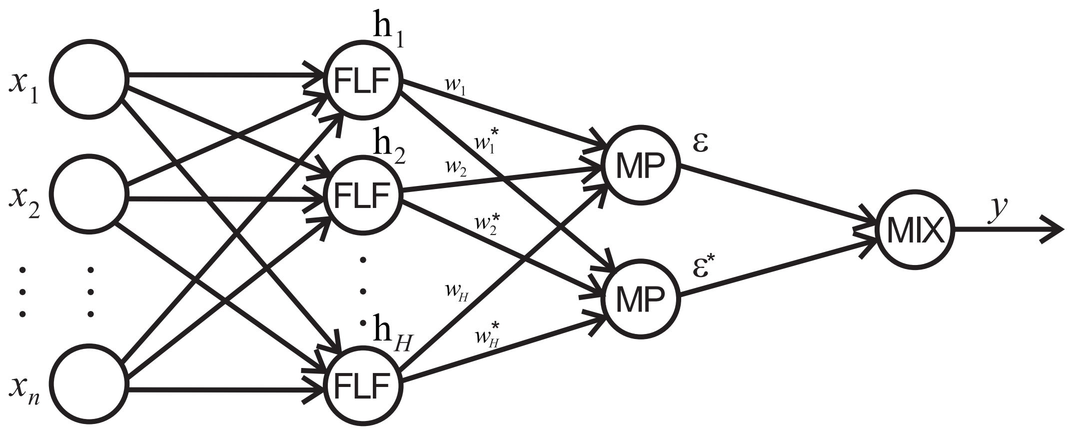

Figure 4: Modus ponens fuzzy network for $m = 1$

The MPFN is able to learn from the pattern set but more sophisticated approach is based on weight optimization.

MPFN Processing Rule

$$\begin{aligned}
&\mathrm{h}_j = \mathrm{FLF}_j(\boldsymbol{x}) && \text{for } j = 1, \dots, H, \\
&\mathcal{E}_i(\boldsymbol{x}) = \bigvee_{j=1}^{H} w_{i,j} \otimes \mathrm{h}_j(\boldsymbol{x}) && \text{for } i = 1, \dots, m, \\
&\mathcal{E}_i^*(\boldsymbol{x}) = \bigvee_{j=1}^{H} w_{i,j}^* \otimes \neg\mathrm{h}_j(\boldsymbol{x}) && \text{for } i = 1, \dots, m, \\
&\mathrm{y}_i(\boldsymbol{x}) = \mathrm{sqrt}(\mathcal{E}_i(\boldsymbol{x})) \otimes \mathrm{sqrt}(\neg\mathcal{E}_i^*(\boldsymbol{x})) && \text{for } i = 1, \dots, m.
\end{aligned}$$

THEOREM: Let $H \in \mathbf{N}$ be number of hidden FLFs in given MPFN. Let λ_j be their sensitivities for $j = 1, \dots, H$. Then any MPFN output is FLF of input variables with the sensitivity

$$\lambda \leq \max_{j=1,\dots,H} \lambda_j .$$

Min-Max Fuzzy Network

The ***min-max fuzzy network*** (MMFN) is a four-layered network which is based on fuzzy logic functions and operators of ŁA_{sqrt}. The first layer consists of $n \in \mathbf{N}$ inputs $x_1, \ldots, x_n \in \mathbf{L}$. The second and third layers are hidden, the first of them contains $F \in \mathbf{N}$ selected fuzzy logic functions $\varphi_1, \ldots, \varphi_F$ and the second one consists of $H \in \mathbf{N}$ neurons $p_1, \ldots, p_H$. Each neuron p_i is a subset of $\{\varphi_1, \ldots, \varphi_F\}$ and it produces the conjunction of its inputs. The fourth layer of MMFN produces one output y using disjunction of the third layer.

From the mathematical point of view, the output of MMFN is described as

$$y = \bigvee_{k=1}^{H} \left(\bigwedge_{j \in p_k} \varphi_j(\boldsymbol{x}) \right) \quad \text{where} \quad p_k \subseteq \{\varphi_1, \ldots, \varphi_F\}.$$

The number H is called a *complexity* of MMFN. It is clearly, the maximal H is 2^F. The selection of H subsets can be a subject of discrete optimization.

The MMFN structure for $n = 9$, $F = 4$, $H = 5$ where $p_1 = \{\varphi_1, \varphi_2\}$, $p_2 = \{\varphi_1, \varphi_3\}$, $p_3 = \{\varphi_2, \varphi_3\}$, $p_4 = \{\varphi_1, \varphi_4\}$ and $p_5 = \{\varphi_2, \varphi_4\}$ is depicted in the Fig. 5.

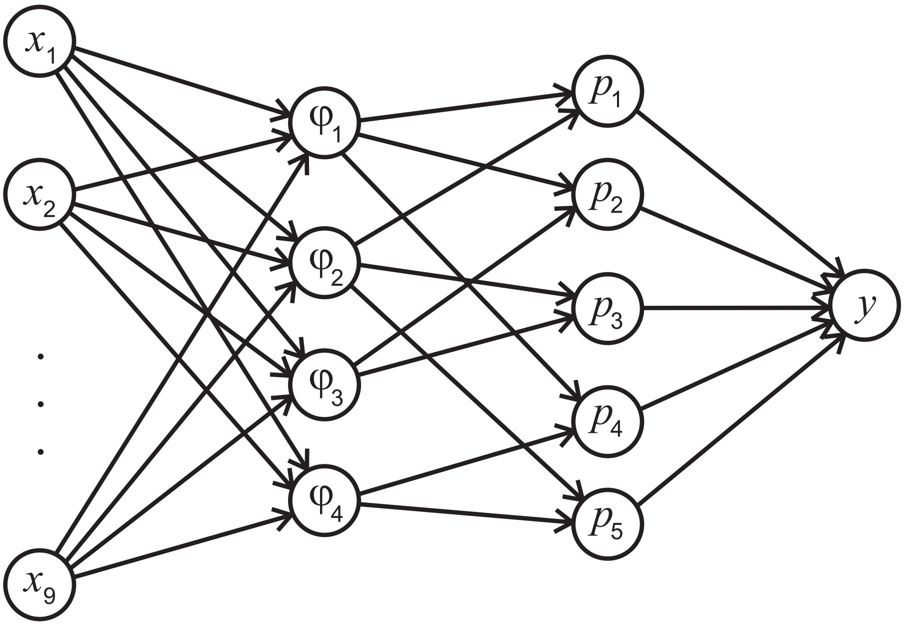

Figure 5: Example of MMFN structure ($n = 9$, $F = 4$, $H = 5$)

If we know the ideal output, we can optimize the MMFN structure—a learning of MMFN is converted to finding the optimum vector of parameters

$$\boldsymbol{p}_{\text{opt}} = \arg\min_{\boldsymbol{p}} \mathrm{f}(\boldsymbol{p}).$$

The MMFN output is a compromise function of individual inputs $x_1, \ldots, x_n$, i.e. it satisfies

$$\min_{k=1,\ldots,n} x_k \leq \mathrm{y}(x_1, \ldots, x_n) \leq \max_{k=1,\ldots,n} x_k.$$

Modular FLF Network

Modular networks consist of N independent networks (subnetworks) and a *gating network*, which determines how much of each network's output is applied to the final output. The single network is also called an *expert network*. The output of modular network is computed as weighted compromise (frequently as weighted sum of subnetworks outputs).

DEFINITION: Let $x, y \in \mathbf{L}$ be values of given property and its ideal. Let $k \in \mathbf{N}$. Then a ***similarity*** of x, y values is defined as FLF function $\mathrm{s} : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$ where

$$\mathrm{s}(x, y) = (x \leftrightarrow y)^k = \max(0, 1 - k \cdot |x - y|).$$

Thus, the similarity $\mathrm{s}(x, y)$ is only a power of the biresiduum. The recommended values are $1 \leq k \leq 1000$. In the case of image processing, there are some useful similarities, for example similarity between filter value and recommended value.

DEFINITION: Let $n, N \in \mathbf{N}$ are number of inputs and number of subnetworks. Let each subnetwork is FLF and y_k be output of k-th subnetwork for $k = 1, \ldots, N$. Let a gating network produces a normalized vector $\boldsymbol{g} \in \mathbf{L}^N$ using similarities of subnetwork outputs with given value x^*. Let $y_{\mathrm{REF}} = \mathrm{r}(\boldsymbol{y})$ where r is a FLF. Then a modular network, whose output

$$y = \mathrm{r}(\boldsymbol{y}) \oplus \left(\bigvee_{k=1}^{N} \left(g_k \otimes (y_k \ominus \mathrm{r}(\boldsymbol{y})) \right) \right) \ominus \left(\bigvee_{k=1}^{N} \left(g_k \otimes (\mathrm{r}(\boldsymbol{y}) \ominus y_k) \right) \right),$$

is called a ***modular FLF network*** (MFLFN). The r is called a ***referential function***.

An example of MFLFN structure is depicted in the Fig. 6.

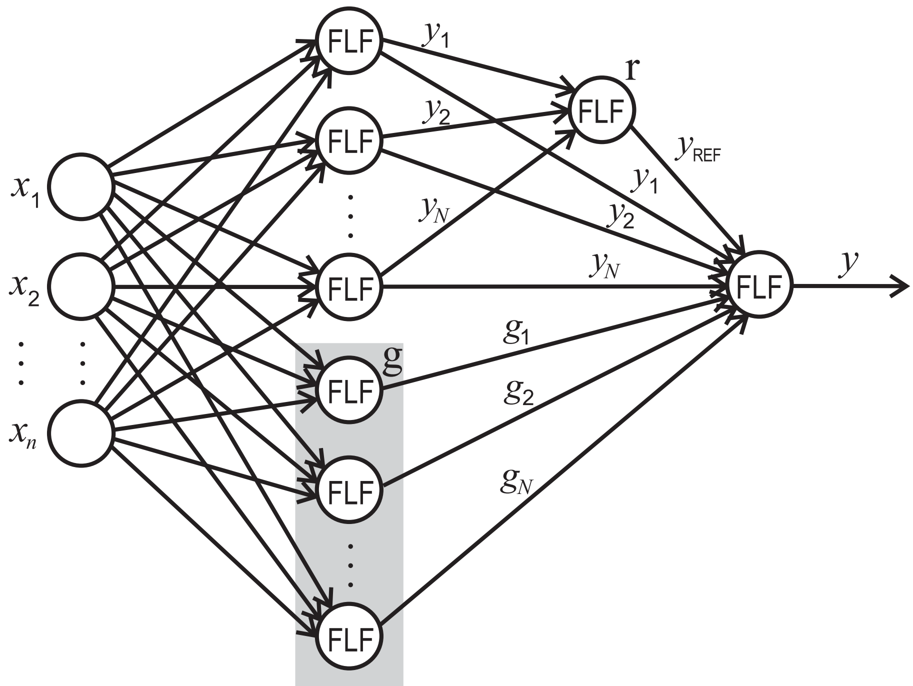

Figure 6: Architecture of MFLFN

THEOREM: MFLFN output is FLF of input variables.

The sensitivity of MFLFN output is also derived.

Constrained Referential Neural Network

It is not prohibited to have a favorite procedure and use it as referential one. The other procedures can represents a traditional approaches to given task and then typically make a frame or acceptable range for the favoured procedure which should not offer the results out of the range.

DEFINITION: Let $n, H \in \mathbf{N}$ be number of inputs and number of standard nodes. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbf{L}^n$, $\boldsymbol{f} = (f_1, \ldots, f_H) \in \mathbf{L}^H$. Let $f_i = \mathrm{f}_i(\boldsymbol{x})$ be a FLF for $i = 1, \ldots, H$. Let $f_\mathrm{R} = \mathrm{f}_\mathrm{REF}(\boldsymbol{x})$, $f_\mathrm{L} = \mathrm{f}_\mathrm{LOW}(\boldsymbol{f})$, $f_\mathrm{U} = \mathrm{f}_\mathrm{UPP}(\boldsymbol{f})$ be FLFs. Let $y = (f_\mathrm{R} \vee f_\mathrm{L}) \wedge f_\mathrm{U}$. Then the structure producing y from $\boldsymbol{x}$ via $\boldsymbol{f}$, f_R, f_L, f_U is called ***constrained referential neural network*** (CRNN).

The structure of CRNN is depicted in the Fig. 7.

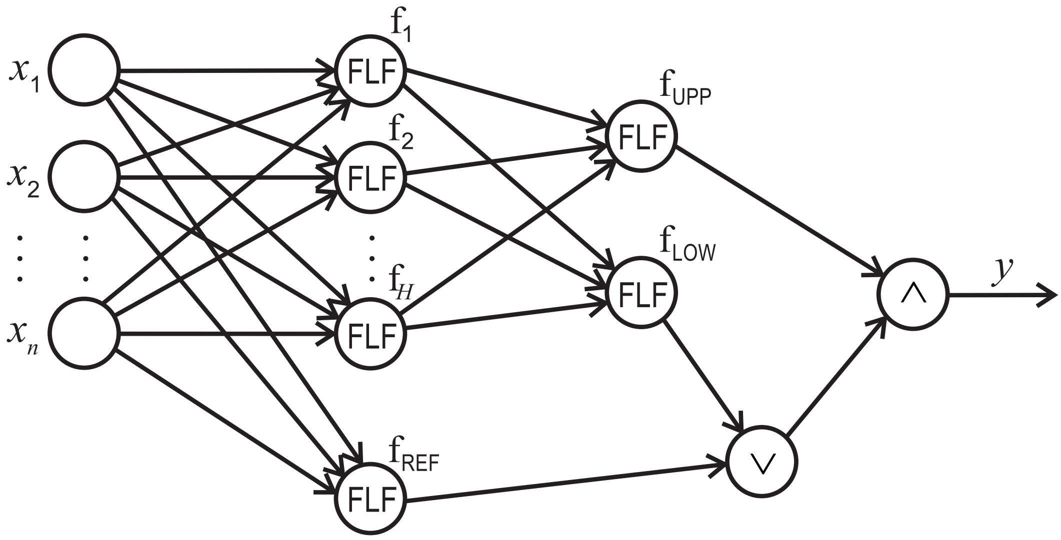

Figure 7: Structure of CRNN

THEOREM: The output y of CRNN is FLF of its input $\boldsymbol{x}$.

The sensitivity of CRNN output is also derived.

CRNN is a special case of fixed MIN-MAX structure applied to f_R, f_L and f_U signals. A trivial but useful range can be obtained by using $f_\mathrm{L} = \bigwedge_{k=1}^{H} f_k$, $f_\mathrm{U} = \bigvee_{k=1}^{H} f_k$. More sophisticated range can be obtained from ordered list $(f_{(1)}, \ldots, f_{(H)})$ using $f_\mathrm{L} = f_{(\mathrm{KEY})}$, $f_\mathrm{U} = f_{(H-\mathrm{KEY}+1)}$ for $\mathrm{KEY} \leq \lfloor H/2 \rfloor$.

Dyadic Weight Neural Network

The last hierarchical approach to image processing is based on a compromise in $\mathrm{ŁA}_\mathrm{sqrt}$ using weighted average with fixed dyadic weights.

DEFINITION: Let $n, H \in \mathbf{N}$ be numbers of input and hidden nodes, $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbf{L^n}$ and $\boldsymbol{f} = (f_1, \ldots, f_H) \in \mathbf{L}^H$. Let $f_i = \mathrm{f}_i(\boldsymbol{x})$ where f_i be FLF for $i = 1, \ldots, H$. Let $m_k, N \in \mathbf{N}_0$, $w_k = m_k/2^N$ and $\sum_{k=1}^{H} w_k = 1$. Let $y = \sum_{k=1}^{H} w_k \mathrm{f}_k(\boldsymbol{x})$ be an output signal. Then the structure producing y from $\boldsymbol{x}$ by $\boldsymbol{f}$ is called ***dyadic weight neural network*** (DWNN).

The structure of DWNN is depicted in the Fig. 8.

The weights can be subject of discrete optimization for fixed exponent N.

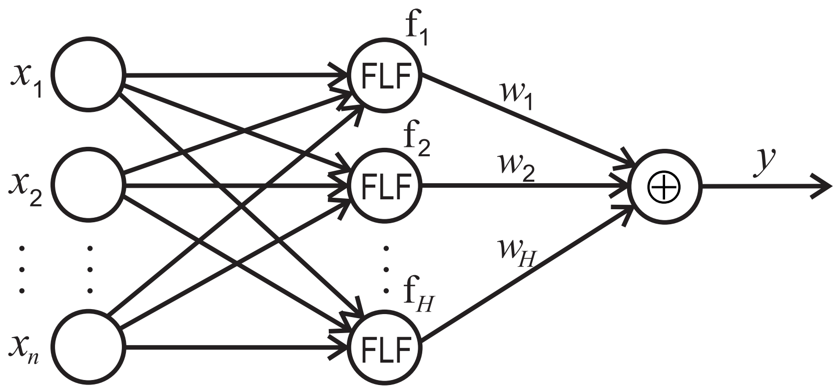

Figure 8: Structure of DWNN

THEOREM: The output of DWNN is FLF of its input $\boldsymbol{x}$.

THEOREM: Let $H \in \mathbf{N}$ and λ_k be a sensitivity of f_k for $k = 1, \ldots, H$. Then the sensitivity of DWNN is $\lambda \leq \max\limits_{1 \leq k \leq H} (\lambda_k)$.

8 Biomedical Image De-Noising

The last part of my doctoral thesis is oriented to biomedical image de-noising. The MRI, PET, CT and SPECT techniques are described first. Then the artificial and ideal technical, MRI and SPECT images are prepared for a testing of fuzzy logic function filters. The artificial technical images were obtained from ideal one applying uniform, gaussian, laplacean and cauchian noise. The artificial biomedical image was obtained using addition of noise from top left corner of real MRI image, while SPECT artificial images were obtained using gaussian noise. Properties of 58 fuzzy logic functions were studied using four quality measures (SNR, MSE, MED, MAE). The best individual filters for MRI image are quasi median ($k = 1$) with binomial mask ($R = 1$) or quasi median ($k = 1$) applied to box mask ($R = 1$) according to SNR or MSE criterion.

The multicriterial approach was applied to the results of individual filtering in the second step. The set of Pareto optimum filters for artificial biomedical images consists of eight filters (Tab. 1). Finally, this set was used in five types of hierarchical networks from previous part. The weights of MPFN and DWNN, the structure of MMFN and the parameters of MFLFN and CRNN were subject of optimization using artificial MRI image with SNR criterion. The results of network optimization are:

- MPFN: $w = (0, 0, 0.5416, 1, 0, 0, 1, 1)$, $w^* = (1, 1, 1, 1, 1, 1, 1, 1)$,
- MMFN: 120 subsets in first hidden layer,
- MFLFN: $k = 2$, r is Hodges–Lehmann median of individual filters, $\boldsymbol{g}$ represents similarity between individual filters and their median,
- CRNN: $\mathrm{f}_{\mathrm{REF}} = F_{13}$, KEY $= 2$,
- DWNN: $w = (0, 17/32, 15/32, 0, 0, 0, 0, 0)$, i.e. the output is affected by F_{13} and F_{14} only.

The quality measures of MPFN, CRNN and DWNN outputs are better than quality of the best individual filter F_{14} on artificial MRI image (Tab. 2). Thus, this three types of networks are recommended for MRI 2D image de-noising.

Filter	**Mask**	1^{st} **level**	2^{nd} **level**
F_1	$\mathbf{M}_1$	M	
F_{13}	$\mathbf{M}_1$	Q_1	
F_{14}	$\mathbf{M}_2$	Q_1	
F_{24}	$\mathbf{M}_2$	Q_2	
F_{27}	$\mathbf{M}_1$	Walsh list	Q_2
F_{28}	$\mathbf{M}_2$	Walsh list	Q_2
F_{33}	$\mathbf{M}_1$	BES	
F_{37}	$\mathbf{M}_1$	Walsh list	BES

Table 1: Selected Pareto optimum filters ($\mathbf{M}_1$: box mask, $\mathbf{M}_2$: binomial mask)

The optimized FLF networks were used for de-noising of real MRI images (human brain and human genu). Selected results are depicted in Figs. 9–20.

Filter	SNR [dB]	MSE	MED	MAE
NOISED	10.8381	0.0671	0.0225	0.0416
F_1	14.6766	0.0414	0.0075	0.0238
F_{13}	15.2065	0.0394	0.0138	0.0248
F_{14}	15.2262	0.0395	0.0138	0.0252
F_{24}	14.9529	0.0416	0.0187	0.0278
F_{27}	14.5299	0.0438	0.0206	0.0299
F_{28}	14.8966	0.0426	0.0212	0.0297
F_{33}	14.7934	0.0416	0.0169	0.0271
F_{37}	14.4287	0.0452	0.0249	0.0325
MPFN	15.3123	0.0395	0.0181	0.0270
MMFN	15.0042	0.0429	0.0200	0.0293
MFLFN	15.1654	0.0401	0.0175	0.0265
CRNN	15.3301	0.0394	0.0169	0.0258
DWNN	15.5375	0.0381	0.0140	0.0245

Table 2: Quality of filtering of artificial MRI image

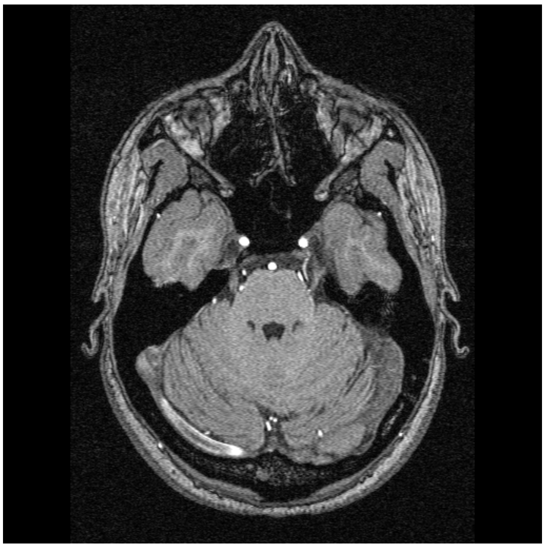

Figure 9: Real MRI image RI_1 (brain)

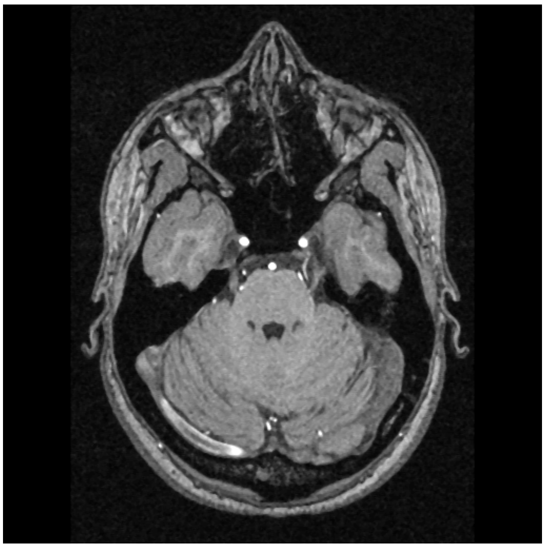

Figure 10: RI_1 filtered by F_{13}

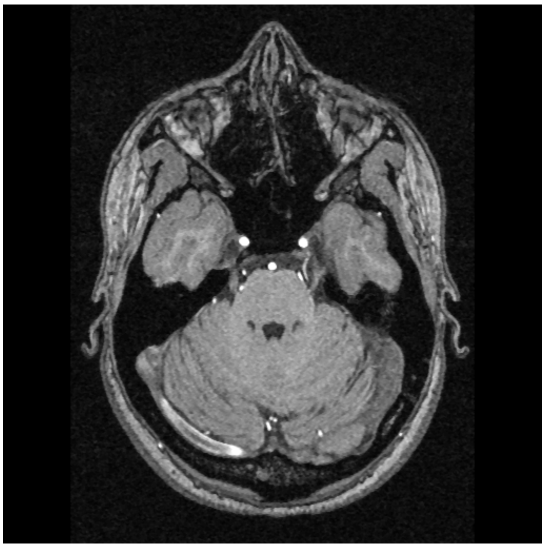

Figure 11: RI_1 filtered by F_{14}

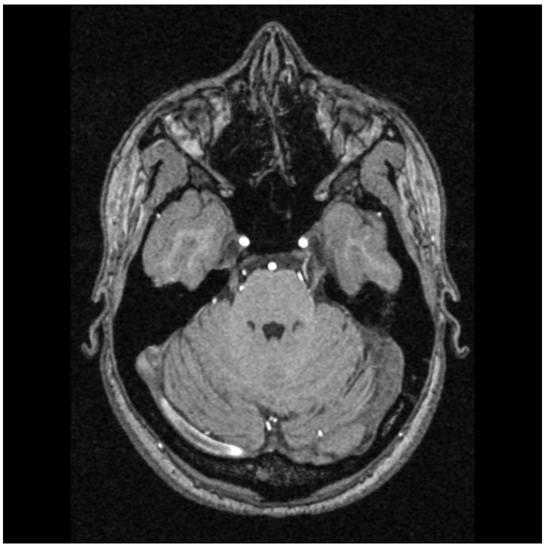

Figure 12: RI_1 filtered by F_{28}

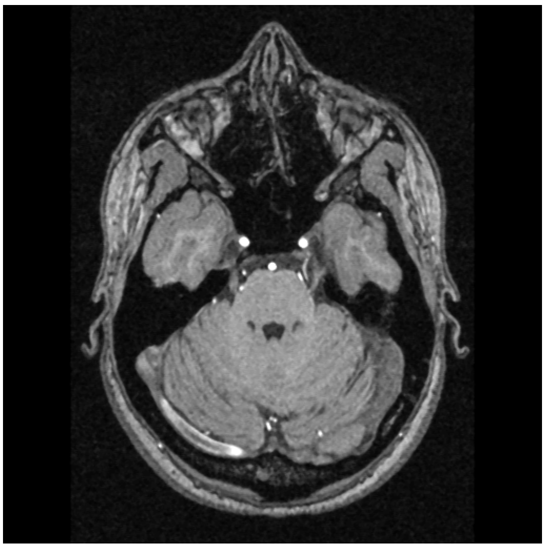

Figure 13: RI_1 filtered by F_{33}

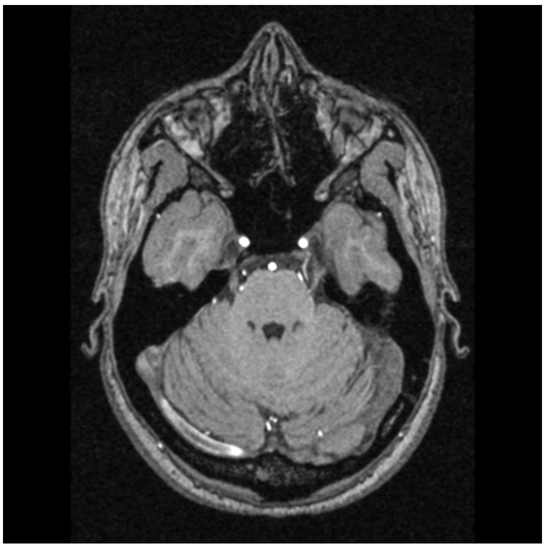

Figure 14: RI_1 filtered by F_{37}

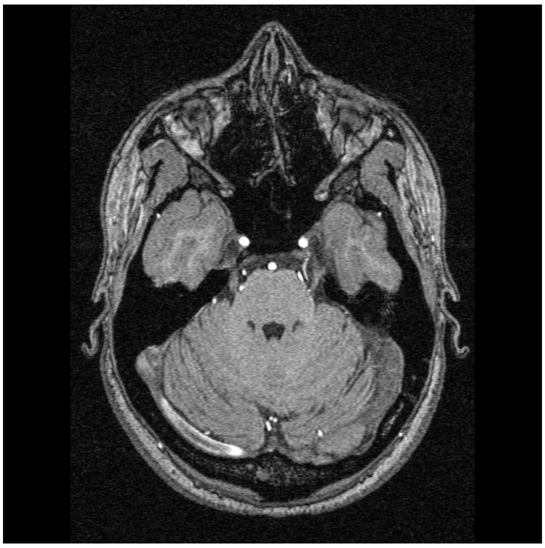

Figure 15: Real MRI image RI_1

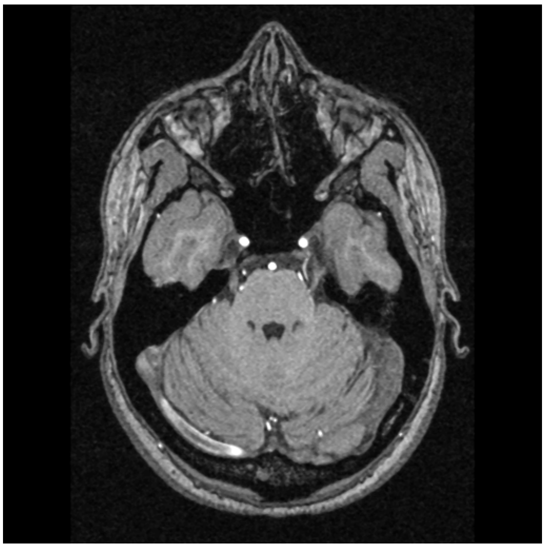

Figure 16: RI_1 filtered by MPFN

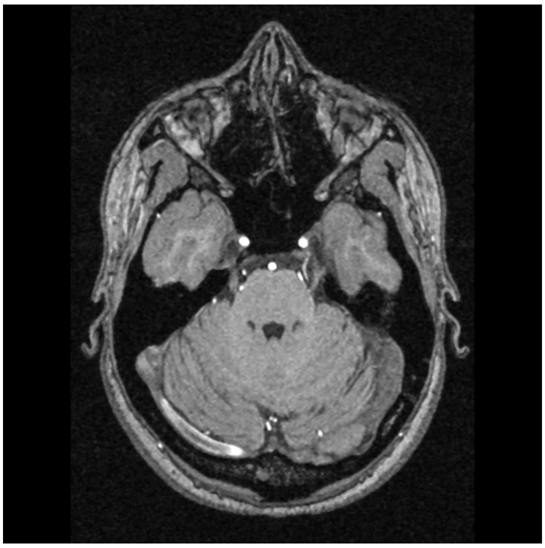

Figure 17: RI_1 filtered by MMFN

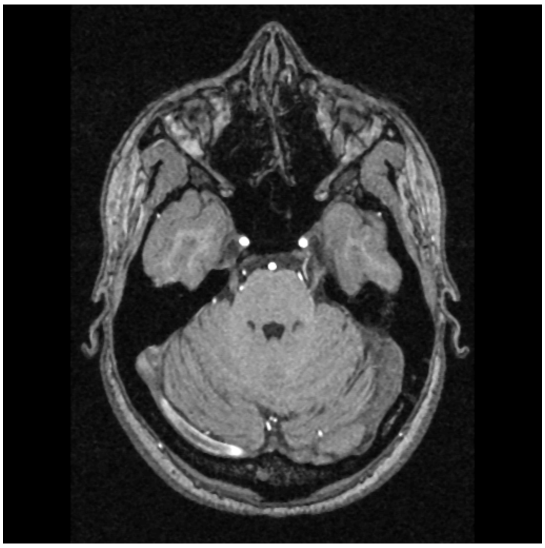

Figure 18: RI_1 filtered by MFLFN

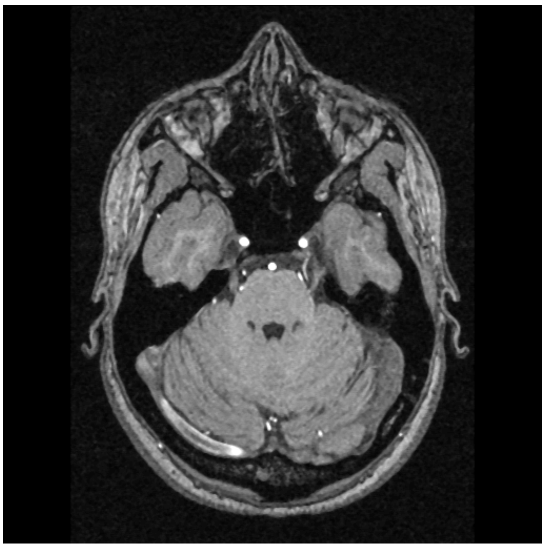

Figure 19: RI_1 filtered by CRNN

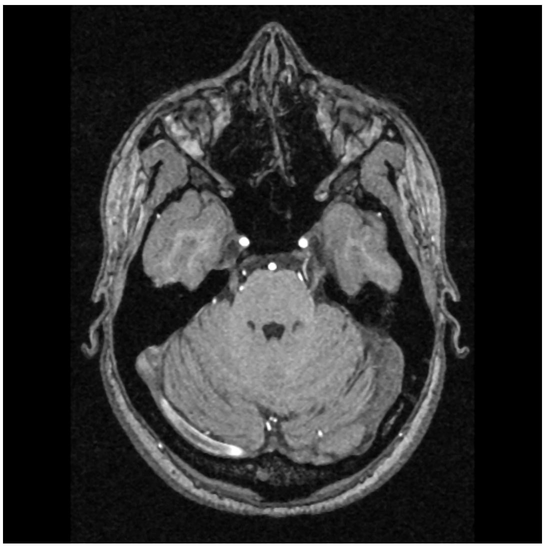

Figure 20: RI_1 filtered by DWNN

Conclusions

Noisy 2D images are typical results of technical monitoring, meteorological observations or biomedical measurements. Noise reduction and structure saving are contradictory but useful aims. Łukasiewicz algebra with square root was used as mathematical model for image processing background. Łukasiewicz algebra with square root as algebraic structure was defined first. This is a simple algebraic model but it has some useful properties:

- the number of operators and functions is minimum possible,
- there is possible to construct infinite number of different expressions and functions,
- the Lipschitz continuity and constrained sensitivity of functions are guaranteed.

Fuzzy logic function and its sensitivity were defined next. Upper bounds of sensitivity of operators and basic functions were inferred. Then the basic terms of 2D image processing were defined and a sample of traditional filters, which are not realizable in Łukasiewicz algebra with square root, was described. The set of fuzzy logic function filters and proofs of their realization in Łukasiewicz algebra with square root were introduced.

The second aim of my work was to define the image enhancement networks as a hierarchical structures of data processing and compare their outputs with results of individual fuzzy logic functions. Łukasiewicz algebra with square root was used for node processing in fuzzy logic function networks. More complex activities can be performed for advanced processing and five network were introduced:

- modus ponens fuzzy network (MPFN),
- min-max fuzzy network (MMFN),
- modular fuzzy logic function network (MFLFN),
- constrained referential neural network (CRNN) and
- dyadic weights neural network (DWNN).

Their sensitivities were also inferred. Finally, the previous hierarchical and individual fuzzy logic function filters were tested and compared on artificial images. The set of them also includes 2D biomedical images with natural and artificial noises. The filtering principles were compared on the basis of traditional quality measures (SNR, MSE, MAE, MED). The multicriterial approach were also used. The best individual and optimized hierarchical fuzzy logic function filters are recommended for the de-noising of biomedical 2D MRI images.

The algorithms of data processing were realized in MATLAB environment and the source code examples are included.

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