Statistical Signal Processing: Shannon, Renyi and Tsallis Entropy for Fractal Set Analysis

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Basic Terms

- **Fractal set**
- **Border Analysis**
- **Topological dimension**
- **Hausdorff dimension**
- **Embedded dimension**
- **Capacity dimension**
- **Information dimension**
- **Correlation dimension**
- **Rényi dimension**
- **Digital image**
- **Estimation of dimensions**

Sample of Brutal Mistakes

- **DFT (FFT) without trend removing**
- **PCA or ICA without removing of mean**
- **MDM without data normalization**
- **Fractal analysis including internal points**
- **Supposing that digital image can be a fractal**
- **Supposing that fractal dimension estimate (from digital image) is unbiased**

Nature of Binary 2D Image

- Model of set $\mathbf{F}$ in $\mathbf{R}^2$
- $\mathbf{F}$ can be a fractal set
- Represented via matrix of 0 and 1
- 1st interpretation: Finite union of squares, then border is a finite union of line segments, $D_{\mathrm{T}} = D_{\mathrm{H}} = 1$
- 2nd interpretation: Finite set of points, then border is the same and then $D_{\mathrm{T}} = D_{\mathrm{H}} = 0$
- Our model is not a fractal set
- Nothing better is available
- IT notice: Increase the image resolution
- Math notice: Do not analyze discrete image
- Optimistic/opportunistic notice: Do it carefully

Enemies of Fractal Analysis

- Space discretization
- Image dimension
- Small image with low resolution
- Large image with high resolution
- Sensitivity to threshold level
- Sensitivity to noise
- Sensitivity to denoising
- Sensitivity to color model

Scheme of Fractal Analysis

- **Image preprocessing**
- **Conversion to binary image (1D, 2D, 3D)**
- **Removing of internal points**
- **Grid application**
- **Counting of points and probabilities**
- **Evaluation of entropy**
- **Averaging of entropy**
- **Model** $\mathrm{H}_q(\varepsilon) = A_q - D_q \log_2 \varepsilon$
- **Method of least squares**

Entropy according to Shannon

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$$

$$\forall k : p_k \geq 0$$

$$\sum_{k=1}^{n} p_k = 1$$

$$H = -\sum_{k=1}^{n} p_k \log_2 p_k$$

Naive entropy for $\varepsilon = 1$

1							
				1	1		
				1			
	1		1				
		1				1	

1/8							
				1/8	1/8		
				1/8			
	1/8		1/8				
		1/8				1/8	

$$H = \log_2 8 = 3.0000$$

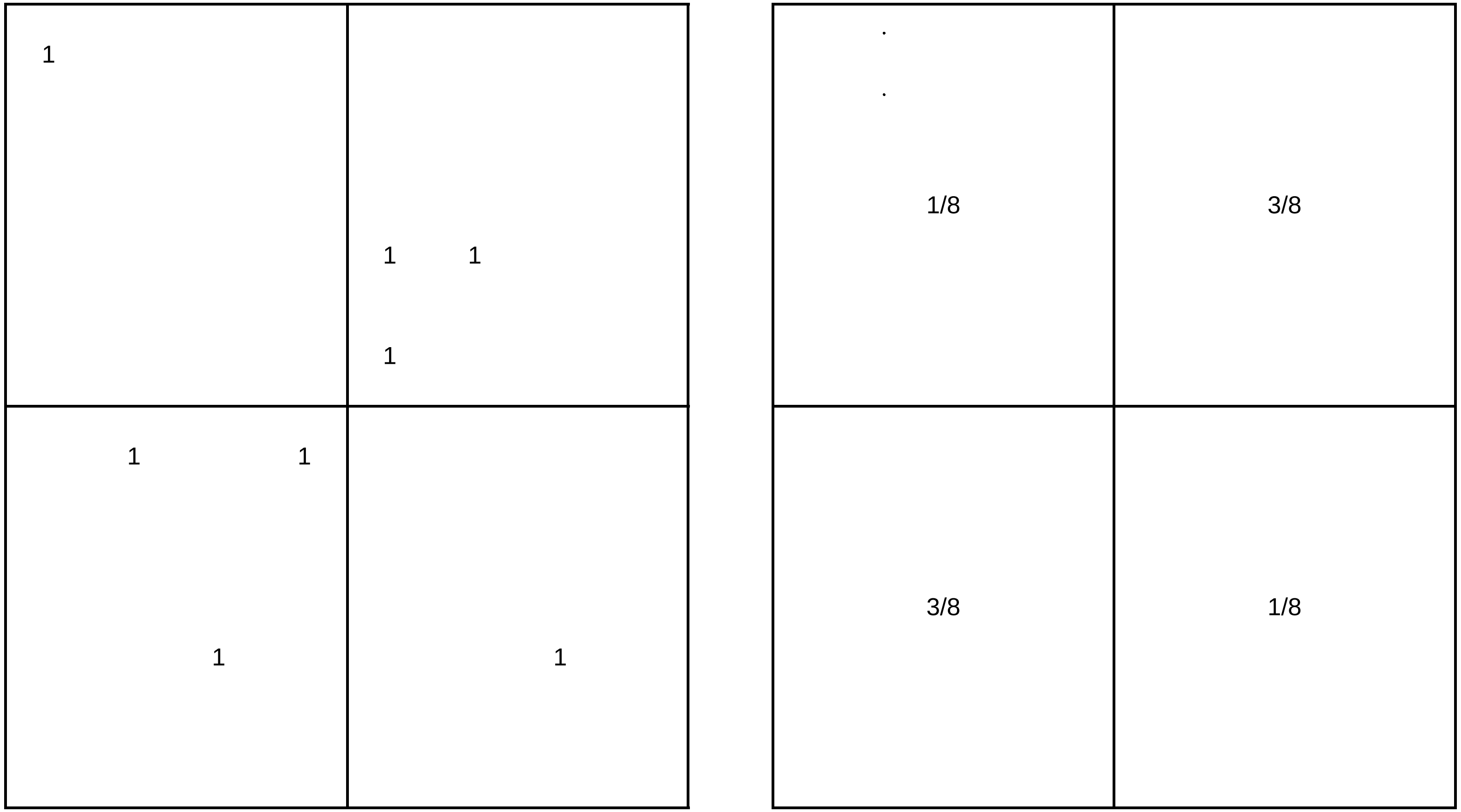
Naive entropy for $\varepsilon = 2$

1			
		1 1 1	
1	1		
	1		1

1/8			
		3/8	
1/8	1/8		
	1/8		1/8

$$H = \log_2 8 - \frac{3}{8}\log_2 3 = 2.4056$$

Naive entropy for $\varepsilon = 4$



$$H = \log_2 8 - \frac{3}{4}\log_2 3 = 1.8112$$

Information dimension

$$D = -\lim_{\varepsilon \to 0+} \frac{\mathrm{d}\, H}{\mathrm{d}\log_2 \varepsilon} \le D_{\mathrm{H}}$$

$$p_k = N_k / N$$

$$H = -\sum_{p_k > 0} p_k \log_2 p_k$$

$$0 \le H \le \log_2 N$$

$$H \approx A - D \log_2 \varepsilon$$

Entropy according to Rényi

$$H_q = \frac{1}{1-q} \log_2 \left(\sum_{p_k > 0} p_k^q \right)$$

$$H_1 = \lim_{q \to 1} H_q = -\sum_{k=1}^{n} p_k \log_2 p_k$$

$$H_{+\infty} = \lim_{q \to +\infty} H_q = -\log_2 p_{\max}$$

$$H_{-\infty} = \lim_{q \to -\infty} H_q = -\log_2 p_{\min}$$

$$H_0 = \log_2 \left(\sum_{p_k > 0} 1 \right)$$

Rényi dimension

$$D_q = -\lim_{\varepsilon \to 0+} \frac{\mathrm{d}\,H_q}{\mathrm{d}\log_2 \varepsilon}$$

$$q < r \Rightarrow D_q \geq D_r$$

$$D_{\mathrm{T}} \leq D_2 \leq D_1 \leq D_{\mathrm{H}} \leq D_0 \leq D_{\mathrm{E}}$$

$$H_q \approx A_q - D_q \log_2 \varepsilon$$

Entropy according to Tsallis

$$T_q = \frac{1}{q-1}\left(1-\sum_{p_k>0} p_k^q\right)$$

$$T_1 = \lim_{q\to 1} T_q = -\sum_{k=1}^{n} p_k \ln p_k = H_1 \ln 2$$

$$T_{+\infty} = \lim_{q\to+\infty} T_q = 0$$

$$T_{-\infty} = \lim_{q\to-\infty} T_q = +\infty$$

$$T_0 = -1 + \sum_{p_k>0} 1$$

Bayessian Approach

- Motivation: Unbiased entropy estimate
- Prior: Uniform distribution of event probabilities
- Constrain: sum of probabilities = 1
- Data: experimental freque ces of events
- Posterior: mean probabilities of events
- Goal: mean entropy

Results for Shannon entropy

$$p_{k,\text{naive}} = \frac{n_k}{N}, \quad N = \sum_{k=1}^{n} n_k$$

$$\mathrm{E}\, p_k = \frac{n_k + 1}{N + n}$$

$$H_\text{naive} = \frac{1}{\ln 2} \sum_{k=1}^{n} \frac{n_k}{N} \ln \frac{N}{n_k}$$

$$H_\text{near} = \frac{1}{\ln 2} \sum_{k=1}^{n} \frac{n_k + 1}{N + n} \ln \frac{N}{n_k}$$

$$\mathrm{E}\, H = \frac{1}{\ln 2} \sum_{k=1}^{n} \frac{n_k + 1}{N + n} \sum_{j=2}^{N+n-n_k} \frac{1}{n_k + j}$$

Removing of internal points

```
function B=MINKOWSKIBORDER(X)
s=size(X);N=sum(s>1);
Y=X;X=zeros(s+2*(s>1));
switch N
    case 1
        if s(1)>1
            MASK=[1;1;1];
            X(2:end-1,:)=Y;
        else
            MASK=[1 1 1];
            X(:,2:end-1)=Y;
        end
    case 2
        MASK=[0 1 0;1 1 1;0 1 0];
        X(2:end-1,2:end-1)=Y;
    case 3
        MASK=zeros(3,3,3);
        MASK(2,2,1)=1;MASK(2,2,3)=1;
        MASK(:,:,2)=[0 1 0;1 1 1;0 1 0];
        X(2:end-1,2:end-1,2:end-1)=Y;
end
B=X-imerode(X,MASK);
```

Probabilities on grid

```
function p=RENYICOVER3D(X,e,si,sj,sk)
[m,n,h]=size(X);
M=ceil((m+si)/e);
N=ceil((n+sj)/e);
H=ceil((h+sk)/e);
X=X/sum(X(:)); P=zeros(M,N,H);
for i=1:M
    imin=max(1,1+(i-1)*e-si);
    imax=min(m,i*e-si);
    for j=1:N
        jmin=max(1,1+(j-1)*e-sj);
        jmax=min(n,j*e-sj);
        for k=1:H
            kmin=max(1,1+(k-1)*e-sk);
            kmax=min(h,k*e-sk);
            B=X(imin:imax,jmin:jmax,kmin:kmax);
            P(i,j,k)=sum(B(:));
        end
    end
end
p=P(:)'; p=p(p>0);
```

Rényi entropy on grid

```
function h=RENYIENTRO3D(X,e,si,sj,sk,q)
n=length(q); h=0*q;
p=RENYICOVER3D(X,e,si,sj,sk);
for k=1:n
    if q(k)==1
        h(k)=-sum(p.*log(p));
    elseif q(k)==inf
        h(k)=-log(max(p));
    elseif q(k)==-inf
        h(k)=-log(min(p));
    else
        h(k)=log(sum(p.^q(k)))/(1-q(k));
    end
end
h=h/log(2);
```

Entropy averaging

```
function [eh,sh]=RENYIEH3D(X,e,q)
[m,n,h]=size(X);
if m>1
    m=e;
end
if n>1
    n=e;
end
if h>1
    h=e;
end
M=m*n*h; N=length(q); H=zeros(M,N);
k=1;
for si=0:m-1
    for sj=0:n-1
        for sk=0:h-1
            H(k,:)=RENYIENTRO3D(X,e,si,sj,sk,q);
            k=k+1;
        end
    end
end
eh=mean(H); sh=std(H)/sqrt(M);
```

Role of grid resolution

```
function [EH,SH]=RENYIHQE3D(X,e,q)
m=length(e);
n=length(q);
EH=zeros(m,n);
SH=EH;
for k=1:m
    [eh,sh]=RENYIEH3D(X,e(k),q);
    EH(k,:)=eh;
    SH(k,:)=sh;
end
```

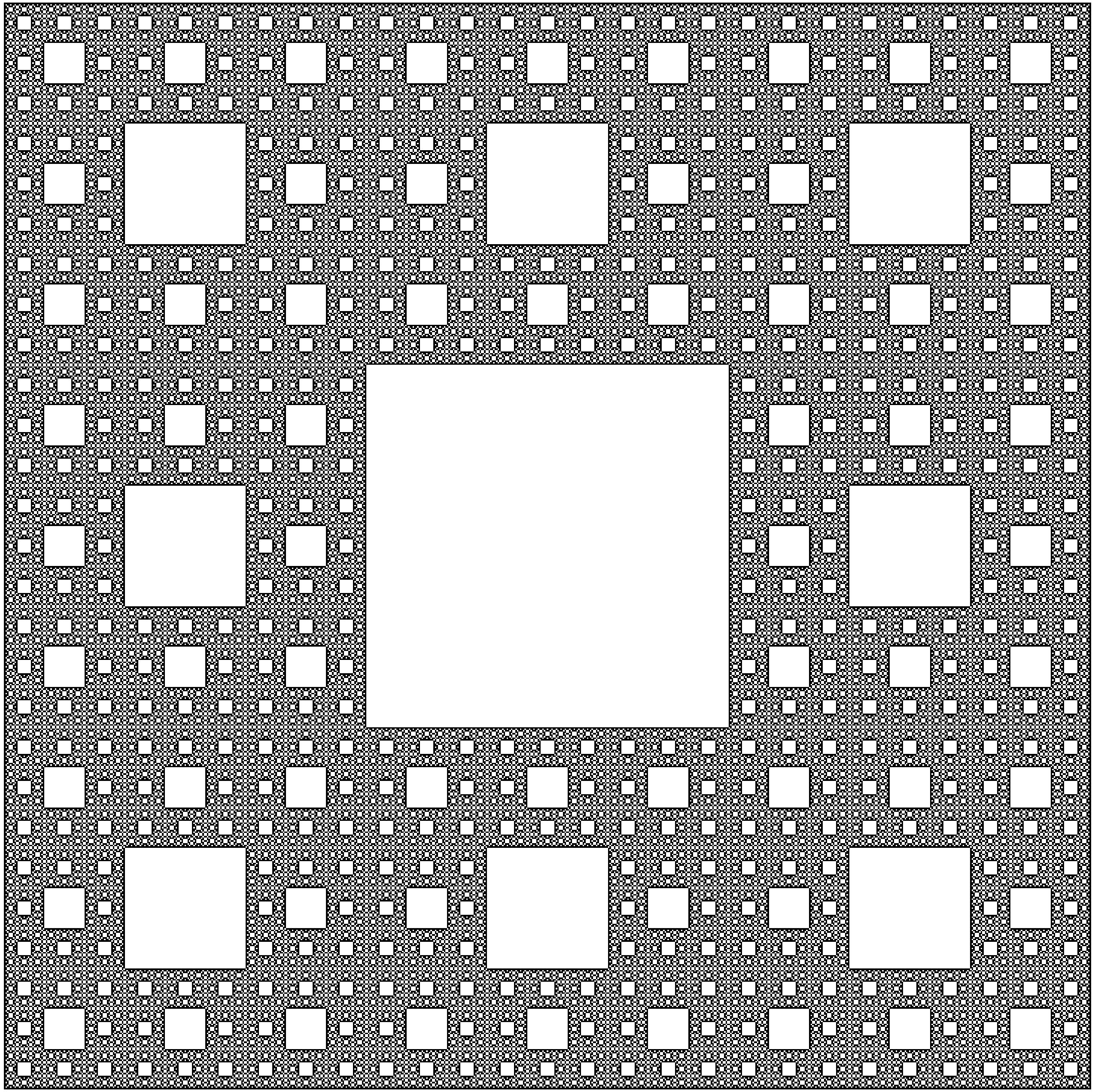
Rényi dimension estimation

```
function [ed,td]=RENYIANALYZER(EH,SH,e)
e=log(e)/log(2); [m,n]=size(EH); SH=max(SH,1e-9);
ed=zeros(1,n);td=ed+inf;
for k=1:n
    X=[ones(m,1)./SH(:,k), e'./SH(:,k)];
    y=EH(:,k)./SH(:,k);
    M=X'*X;
    b=pinv(X)*y; ed(k)=-b(2);
    if m>2 && det(M)~=0
        r=X*b-y;
        se=norm(r)/sqrt(m-2);
        M=inv(M);
        td(k)=se*sqrt(M(2,2))*tinv(1-0.05/2,m-2);
    end
end
```

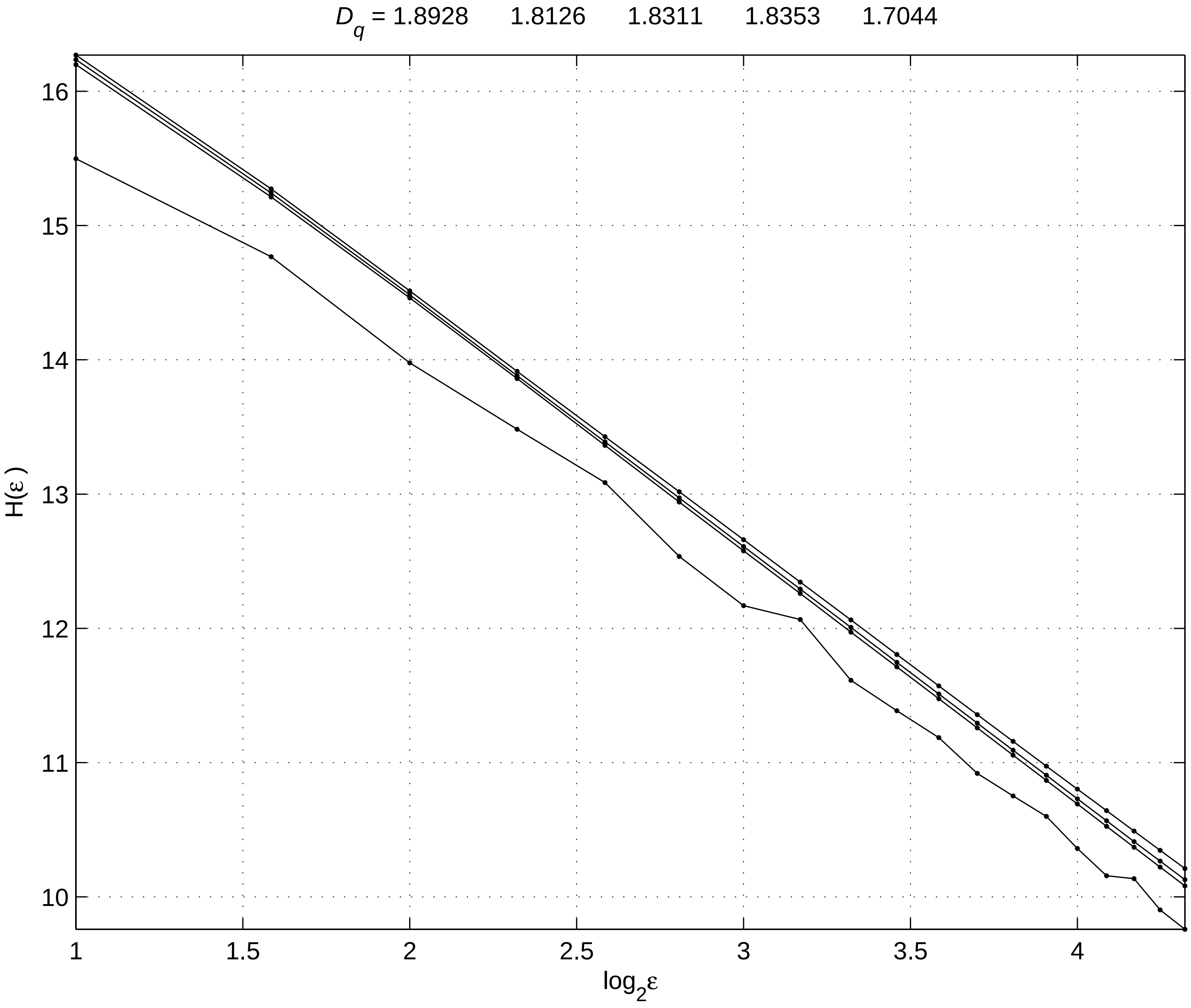
Useful script

```
X=MINKOWSKIBORDER(X);
e=[5 7 11 13 17 19 23 29 31 37 41 43];
q=[0:2, inf];
[EH,SH]=RENYIHQE3D(X,e,q);
[ed,td]=RENYIANALYZER(EH,SH,e);
figure
plot(log(e')/log(2),EH,’k.-’)
xlabel('log_2{\it e}')
ylabel('H({\it e} )')
axis equal
title(['{\it D} = ' num2str(ed) ' \pm ', num2str(td)])
```

Sierpinski carpet in 2D



Entropy changes for D_H, D_0, D_1, D_2 and D_{inf}



Capacity dimension D_0

- Boxcounting method is slow
- Minkowski sausage
- Border dilation
- Distance transform
- Convolution with mask
- Multiplication in Fourier domain
- Application of FFT
- Fast method for large images
- Upper bound of Hausdorff dimension

Correlation dimension D_2

- Sandbox method is slow
- Autocorrelation function
- Convolution
- Multiplication in Fourier domain
- Application of FFT
- Fast method for large images
- Time series analysis
- Lower bound of Hausdorff dimension

Instead of conclusions

$$D_{\mathrm{H}} > D_{\mathrm{T}} \Leftrightarrow \text{fractal}$$

$$D_1 > D_{\mathrm{T}} \Rightarrow \text{fractal}$$

$$D_2 > D_{\mathrm{T}} \Rightarrow \text{fractal}$$

$$D_0 = D_{\mathrm{T}} \Rightarrow \neg\,\text{fractal}$$

$$D_q = \text{const} > D_{\mathrm{T}} \Rightarrow \text{monofractal}$$

$$\text{multifractal} \Leftrightarrow \text{fractal} \wedge \neg\,\text{monofractal}$$