2

Discrete Signals and Systems

Let us start the study of digital signal processing methods by the summary of basic signals and systems properties and mathematical tools enabling their description and analysis. Such a background is substantial for the development of classical and adaptive methods described further.

2.1 Fundamental Concepts

Most of observed *signals* are *continuous functions* $x_a(\mathbf{t})$ of one or more variables. Before their digital processing it is necessary to realize their *sampling* with a given sampling period T_s (or sampling frequency $f_s = 1/T_s$). In case of one independent variable (usually standing for time) resulting *discrete-time signal* is represented (Fig. 2.1) by a sequence of numbers

$$\mathbf{x} = \{x(n)\} = \{x_a(nT_s)\} \tag{2.1}$$

for $n \in (-\infty, +\infty)$. As real analog/digital converters are able to approximate discrete-time values by a limited number of digits only such a sequence is *digital* in fact [30, 23].

Time domain signal description enables definition of *deterministic signals* including *periodic* and *nonperiodic signals* by their mathematical definition. The most important deterministic signals represent

- unit sample sequence: $d(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$
- unit step sequence: $u(n) = \begin{cases} 1 & \text{for } n \geq n_0 \\ 0 & \text{for } n < n_0 \end{cases}$
- real exponential sequence: $x(n) = a^n$
- sinusoidal sequence: $x(n) = A\, sin(2\pi f n)$

The sketch of these signals is given in Fig. 2.2.

Further signals may be described by their own mathematical definition and they may be also combined using the basic operations summarized in Tab. 2.1 (including also MATLAB notation which in real programs does have no formal difference between scalars, vectors or matrixes considering a scalar as a special matrix with one element only).

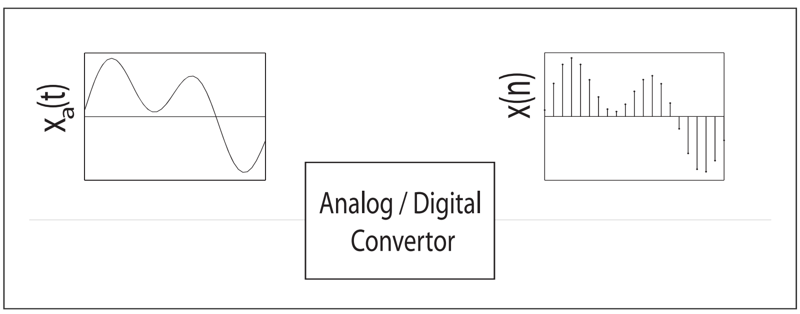

FIGURE 2.1. Sampling process of an analog signal

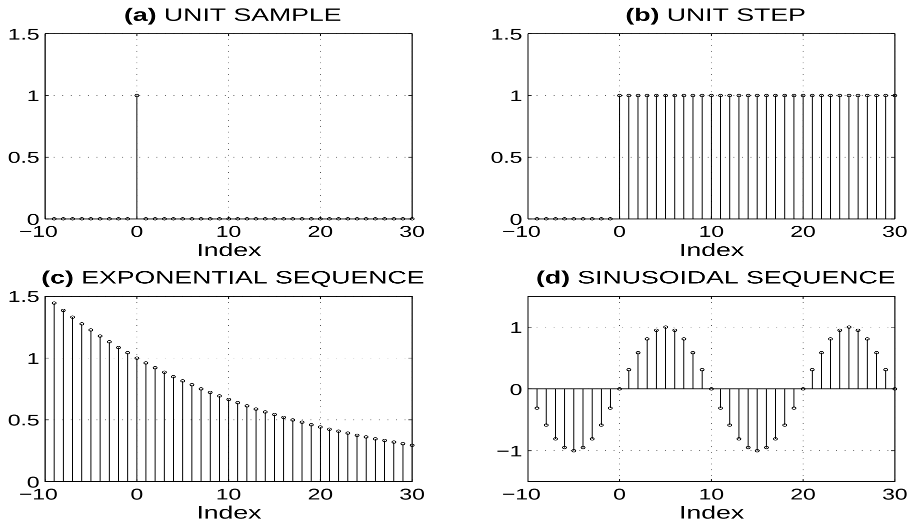

FIGURE 2.2. Basic deterministic signals

In many practical cases observed signals are *random* including unpredictable noise as well. Description of such signals is based upon a random signal theory presented in many books including [31, 2, 12]. The analysis of these signals may be in many cases restricted to *stationary random signals* with their basic probabilistic characteristics (average and autocovariance function) independent of the starting index of observation. An example of a random signal with its histogram approximating its probabilistic distribution is given in Fig. 2.3.

For various signal analysis techniques it is useful to refer to the *energy* of a sequence defined ([23, p.24] or [30, p.10]) as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.2}$$

Frequency domain signal description is another method of the given sequence approximation which is substantial in many digital signal processing methods. Fourier series applied for continuous signals studied for example in [24, p.10] or [23, p.258] are originally restricted to the approximation of a periodic function $f(t)$ with period T by the weighted sum of complex exponentials or trigonometric functions in the form

$$f_{aprox}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\frac{2\pi}{T}t} = a_0 + \sum_{k=1}^{\infty}\left(a_k cos(k\frac{2\pi}{T}t) + b_k sin(k\frac{2\pi}{T}t)\right) \tag{2.3}$$

Operation	*Definition*	*MATLAB notation*
multiplication	$\{x(n)\} \cdot \{y(n)\} \equiv \{x(n) \cdot y(n)\}$	$\mathbf{x} .\star \mathbf{y}$
linear combination	$a \cdot \{x(n)\} + b \cdot \{y(n)\} \equiv \{a \cdot x(n) + b \cdot y(n)\}$	$a\star\mathbf{x}+b\star\mathbf{y}$
convolution	$\{x(n)\} \star \{y(n)\} \equiv \{\sum_{k=-\infty}^{\infty} x(k)y(n-k)\}$	conv($\mathbf{x}$,$\mathbf{y}$)
translation	$\{x(n-n_0)\}$	

TABLE 2.1. Basic sequence operations

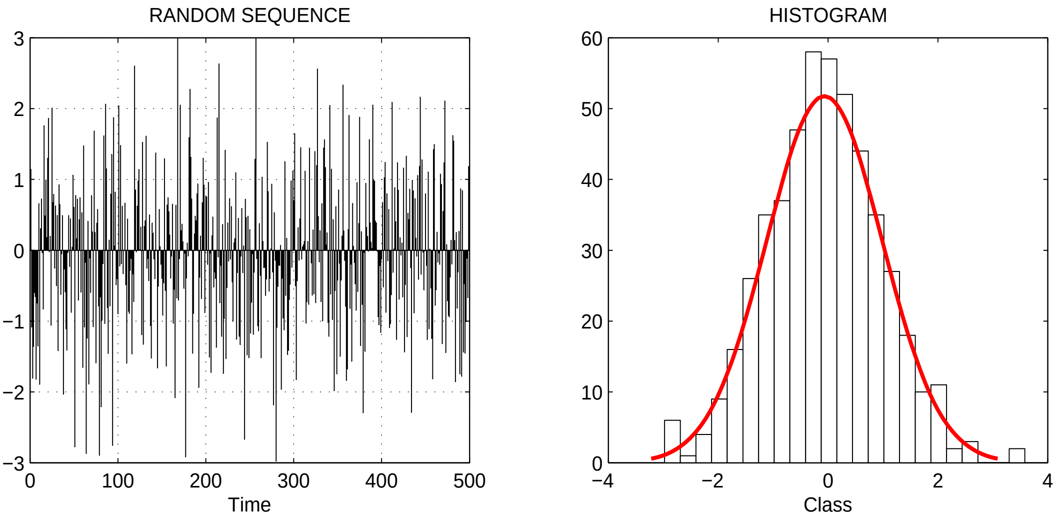

FIGURE 2.3. Stationary random signal with normal probabilistic distribution and its histogram

Using the mean square error method it is possible to derive that

$$F_k = \frac{1}{T} \int_0^T e^{-jk\frac{2\pi}{T}t}\, dt \tag{2.4}$$

for $k = 0, \pm 1, \pm 2, \cdots$ and after the application of Euler relations for complex exponentials it is possible to express

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t)\, dt \\ a_k &= \frac{1}{T} \int_0^T f(t)\, cos(k\frac{2\pi}{T}t)\, dt \\ b_k &= \frac{1}{T} \int_0^T f(t)\, sin(k\frac{2\pi}{T}t)\, dt \end{aligned} \tag{2.5}$$

for $k = 1, 2, \cdots$ .Example of such an approximation of a rectangular function with its period $T = 2\pi$ by a limited number of terms in the form

$$f_{aprox}(t) = \frac{4}{\pi}(sin(t) + \frac{1}{3}sin(3t) + \frac{1}{5}sin(5t) + \frac{1}{7}sin(7t)) \tag{2.6}$$

is presented in Fig. 2.4 together with weights denoting the significance of separate frequency components. Generalization of this method to non-periodic signals is studied further in connection with the *discrete Fourier transform* analysed for instance in [40, p.59] as well. It enables signal description in the form of a finite number of its frequency components giving possibility of the sampling rate estimation as well.

Theorem 2.1 *Let f_m is the highest frequency component of a signal. Then the sampling frequency f_s must be greater or equal then $2f_m$ to enable its perfect reconstruction.*

Proof of this theorem is closely connected with the theory of the discrete Fourier transform presented further and studied in many books including [30, p.28], [40, p.45] or [23, p.57].

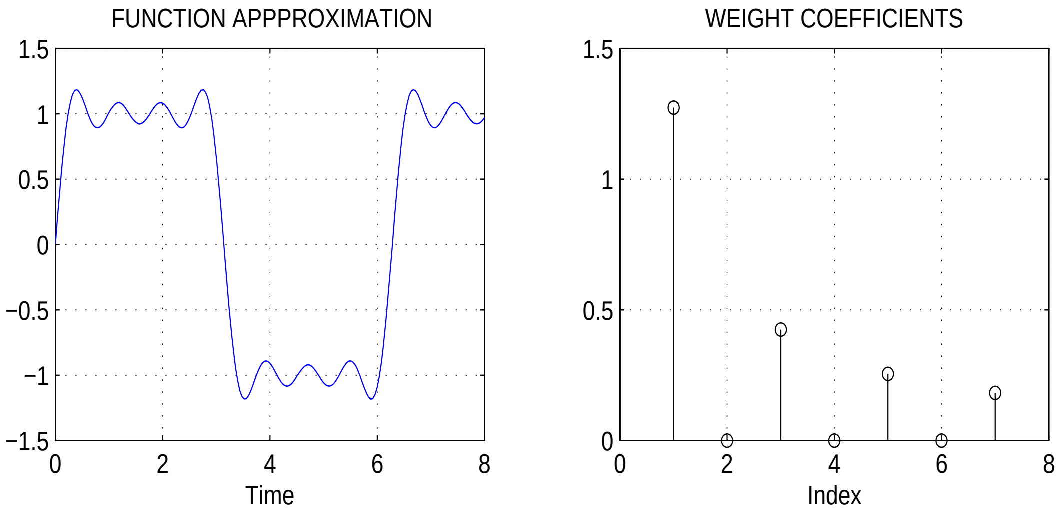

FIGURE 2.4. Rectangular function approximation with the separate frequency components

2.2 Discrete System Description

A *discrete system* is mathematically defined as a transform of the input sequence $\{x(n)\}$ into the output sequence $\{y(n)\}$ by means of an operator $\mathbf{T}$ (Fig. 2.5). In case of the unit sample input sequence $\{d(n)\}$ the system output is called the *impulse response* $\{h(n)\}$ having substantial role in signal analysis presented further. Process of such a transformation is often called *digital filtering* which in the broader sense includes both extraction of information from a given signal and system identification or control as well.

Definition 2.1 *Linear shift invariant system is a discrete system having the following properties*

$$\mathbf{T}[a\, x_1(n) + b\, x_2(n)] = a\, \mathbf{T}[x_1(n)] + b\, \mathbf{T}[x_2(n)] \tag{2.7}$$

$$\mathbf{T}[x(n)] = y(n) \Rightarrow \mathbf{T}[x(n-k)] = y(n-k) \tag{2.8}$$

Theorem 2.2 *Let* $\{h(k) : h(k) = \mathbf{T}[d(k)]\}$ *stands for the impulse response of a discrete linear shift invariant system. Then the response of this system to the signal* $\{x(n)\}$ *is determined by the convolution sum*

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \tag{2.9}$$

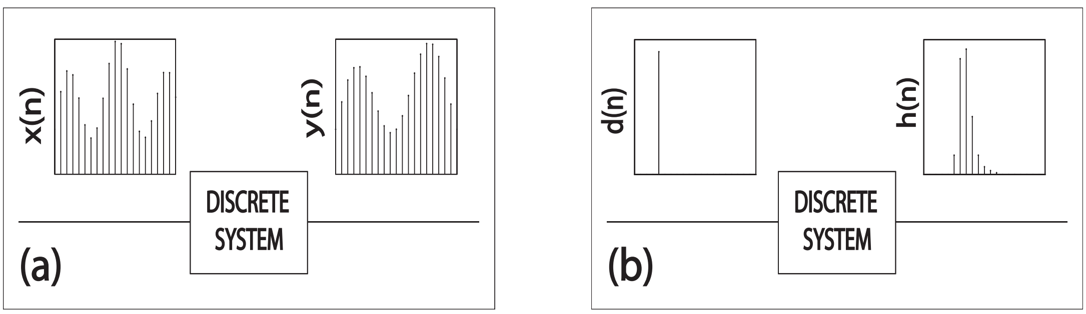

FIGURE 2.5. General discrete system and its application for impulse processing

Proof: It is obvious that it is possible to define signal $\{x(n)\}$ by means of the impulse function $\{d(n)\}$ in the form

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\, d(n-k)$$

Using system operator T it is possible to calculate the system output

$$y(n) = \mathbf{T}[\sum_{k=-\infty}^{\infty} x(k)\, d(n-k)]$$

After application of properties (2.7) and (2.8) of a shift invariant system it is possible to write

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)\, \mathbf{T}[d(n-k)] = \\ &= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \end{aligned}$$

By further substitution to change indices we shall receive expression (2.9). △

Results presented above implies that any linear shift invariant system is completely defined by its unit sample response $\{h(n)\}$. This result can be further used to determine *system stability* [23, p.34].

Definition 2.2 *A discrete system is said to be stable if every bounded input sequence $\{x(n)\}$ implies bounded output sequence $\{y(n)\}$.*

Theorem 2.3 *A linear shift invariant system is stable if and only if the sum*

$$S = \sum_{k=-\infty}^{\infty} |h(k)| \tag{2.10}$$

has a finite value.

Proof: Assume that the input sequence is bounded by a finite M such that $|x(n)| < M$ for all n. Then it is possible to use Eq. (2.9) and the triangular inequality to write

$$|y(n)| = |\sum_{k=-\infty}^{\infty} h(k)\, x(n-k)| \le \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n-k)| < M \sum_{k=-\infty}^{\infty} |h(k)| = M\, S$$

Therefore if S is finite the output sequence is bounded as well. △

Further considerations are in most cases restricted to *causal systems* [23, p.38] having their output for each n dependent on input values for $k \le n$ only. Impuls response $\{h(n)\}$ of such systems is nonzero for $n \ge 0$ only and Eq. (2.9) has therefore the following form

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \tag{2.11}$$

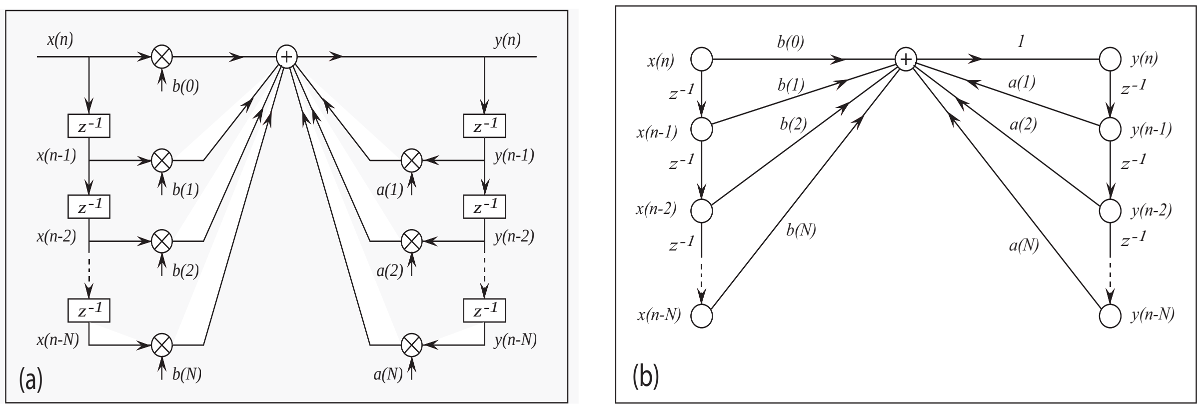

FIGURE 2.6. Block diagram and signal flow graf representation of the IIR filter (with symbol z^{-1} standing for the unit sample delay)

Time domain discrete system description may be in many cases restricted to the *linear constant coefficient difference equation* ([40, p.180], [30, p.16]) defining relationship between the input and output sequence in the form

$$y(n) + \sum_{k=1}^{N} a(k)\, y(n-k) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{2.12}$$

This general equation denoted as *autoregressive-moving average* (ARMA) *model* can take the following specific simplifications

(i) *moving average* (MA) *model* in the form

$$y(n) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{2.13}$$

(ii) *autoregressive* (AR) *model* in the form

$$y(n) + \sum_{k=1}^{N} a(k)\, y(n-k) = x(n) \tag{2.14}$$

Comparing Eqs. (2.13) and (2.11) it is possible to see that coefficients $\{b_0, \cdots, b_N\}$ stand for the finite duration impulse response $\{h_0, \cdots, h_N\}$ and corresponding digital system is therefore also called *finite impulse response* (FIR) *filter*. Owing to Theorem 2.2 it is always stable which explains one of reasons of its popularity.

General autoregressive and autoregressive-moving average model represent *infinite impulse response* (IIR) *filter* as explained in the following example. Graphical description of such a general system in the *block diagram* form and *signal flow graf representation* [30, p.136] is presented in Fig. 2.6.

Example 2.1 Calculate the unit sample response of a digital system described by the difference equation (2.12).

Solution: Assume the input sequence $\{x(n)\} = \{d(n)\}$. Denoting the impulse response $\{y(n)\} = \{h(n)\}$ it is possible to use Eq. (2.12) to evaluate sequence $\{h(n)\}$ in the form

$$\begin{aligned}
h(n) &= 0, \qquad n < 0 \\
h(0) &= b_0 \\
h(1) &= b_1 - a_1\, h(0) \\
h(2) &= b_2 - a_1\, h(1) - a_2\, h(0) \\
&\cdots \\
h(N) &= b_N - \sum_{k=1}^{N} a_k\, h(N-k) \\
h(n) &= -\sum_{k=1}^{N} a_k\, h(n-k), \qquad n > N
\end{aligned}$$

This generally infinite sequence stands for the infinite impulse response of studied digital system.

General shift invariant model of the linear shift invariant system described by the difference Eq. (2.12) can be expressed in the following vector form

$$y(n) + \mathbf{a} \begin{bmatrix} y(n-1) \\ \vdots \\ y(n-N) \end{bmatrix} = \mathbf{b} \begin{bmatrix} x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \tag{2.15}$$

where $\mathbf{a}=[a_1, \cdots, a_N]$
$\mathbf{b}=[b_1, \cdots, b_N]$

This system representation involves calculations with past values of the signal output variables.

State space representation of a digital filter described for instance in [18, p.84] or [23, 231] enables evaluation of the output value $y(n)$ as a linear combination of the input value $x(n)$ and *state variables* $\mathbf{v}\ (n) = [v_1(n), \cdots, v_N(n)]'$ in the form

$$y(n) = \mathbf{c}\, \mathbf{v}(n) + d\, x(n) \tag{2.16}$$

where $\mathbf{c} = [c_1, \cdots, c_N], d$ are the state space model coefficients. The state space vector of the system represents the minimal information required to determine the output and it must be updated for each n for a linear discrete system by state equation

$$\mathbf{v}(n+1) = \mathbf{A}\, \mathbf{v}(n) + \mathbf{b}\, x(n) \tag{2.17}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ & \cdots & \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \tag{2.18}$$

stand for so called state transition matrix and excitation vector respectively.

State space method can be simply applied for multiple inputs and outputs as well (using vectors and matrices instead of scalars and vectors) and can be also used for time varying systems. Graphical description of a general state space representation of a discrete system in signal flow graf notation is presented in Fig. 2.7.

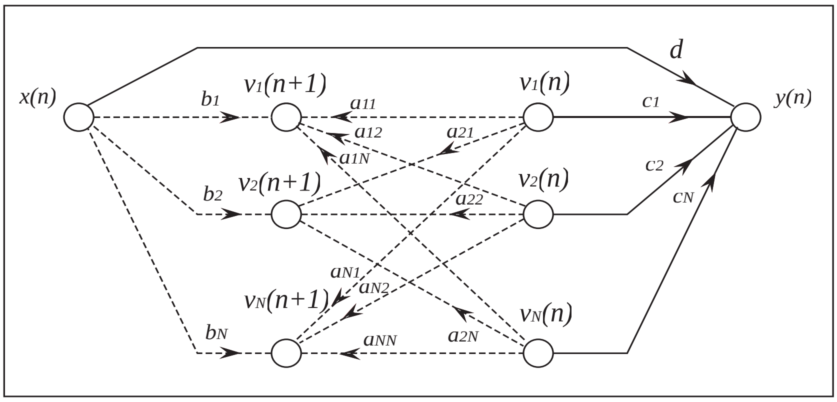

FIGURE 2.7. Signal flow graf representation of the state space system description

Example 2.2 *Derive space equations for the FIR system described by the difference equation in the form*

$$y(n) = h_0\, x(n) + h_1\, x(n-1) + \cdots + h_N\, x(n-N)$$

Solution: Let us define the first state variable

$$v_N(n) = h_1\, x(n-1) + h_2\, x(n-2) + \cdots + h_N\, x(n-N)$$

allowing to express the output equation in the form

$$y(n) = [0\, 0 \cdots 1]\, \mathbf{v}(n) + h_0\, x(n) \tag{2.19}$$

To derive further state equations let us express

$$\begin{aligned} v_N(n+1) &= h_1\, x(n) + h_2\, x(n-1) + \cdots + h_N\, x(n+1-N) = \\ &= h_1\, x(n) + v_{N-1}(n) \end{aligned}$$

with the next state variable in the form

$$v_{N-1}(n) = h_2\, x(n-1) + h_3\, x(n-2) + \cdots + h_N\, x(n+1-N)$$

In the same way it is possible to derive

$$\begin{aligned} v_{N-1}(n+1) &= h_2\, x(n) + v_{N-2}(n) \\ v_{N-2}(n+1) &= h_3\, x(n) + v_{N-3}(n) \\ &\cdots \\ v_2(n+1) &= h_{N-1}\, x(n) + v_1(n) \\ v_1(n+1) &= h_N\, x(n) \end{aligned}$$

and to write the state equation in the form

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} h_N \\ h_{N-1} \\ \vdots \\ h_2 \\ h_1 \end{bmatrix} x(n) \tag{2.20}$$

FREQUENCY DOMAIN REPRESENTATION of a linear shift invariant system ([30, p.19] or [23, p.84]) is very useful in the linear system theory as it provides information about signal processing with respect to its frequency components. In particular the steady state response of such a system to the sinusoidal function is a sinusoid of the same frequency but different amplitude and phase determined by the system.

Since a sinusoid can be defined by the sum of two complex exponentials we can apply a discrete input sequence in the form

$$x(n) = e^{j\omega n}$$

Using Theorem 2.2 it is possible to determine system output in the form

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

Defining the *frequency response*

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

we can evaluate the system output

$$y(n) = H(e^{j\omega})\, e^{j\omega n} \tag{2.21}$$

Using magnitude and phase of the frequency response it is further given by expression

$$y(n) = |H(e^{j\omega})|\, e^{arg(H(e^{j\omega}))}\, e^{j\omega n}$$

Result presented by Eq. (2.21) is valid under assumption that the input sequence has been applied for $k \to -\infty$. In real applications the discrete time system provides *transient period* before the *steady state* response.

Example 2.3 *Evaluate amplitude frequency response of a moving average system described by equation*

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

Solution: Applying $x(n) = e^{j\omega n}$ we shall receive

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\omega(n-k)} = e^{j\omega n} \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\omega k}$$

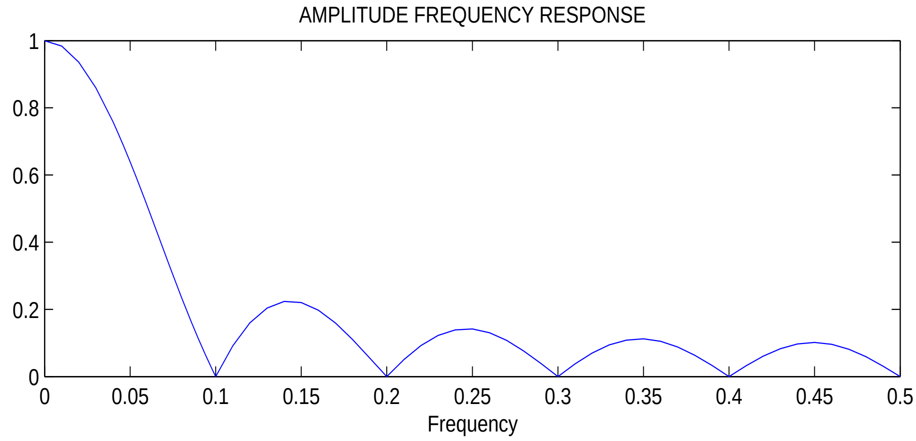

FIGURE 2.8. Amplitude frequency response of the moving average discrete system

Evaluating the sum of the geometrical sequence we shall obtain

$$y(n) = e^{j\omega n} \frac{1}{N} \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

As

$$H(e^{j\omega}) = \frac{1}{N} \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{1}{N} \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

we can evaluate its amplitude using Euler relations in the form

$$\begin{aligned} \left| H(e^{j\omega}) \right| &= \frac{1}{N} \left| \frac{cos(\omega N/2) + jsin(\omega N/2) - (cos(\omega N/2) - jsin(\omega N/2))}{cos(\omega/2) + jsin(\omega/2) - (cos(\omega/2) - jsin(\omega/2))} \right| = \\ &= \frac{1}{N} \left| \frac{sin(\omega N/2)}{sin(\omega/2)} \right| \end{aligned}$$

Amplitude frequency response given in Fig. 2.8 for $\omega \in \langle 0, \pi \rangle$ provides information about the system behaviour with respect to the signal frequency components.

Difference equations or state space representation provide possibilities for the time domain digital system application while the frequency response provides information about its behaviour with respect to its frequency components. Methods of parameter estimation enabling signal analysis or its processing to achieve prescribed system behaviour are studied in next sections.