4

Signal Analysis

Signal analysis is a very useful tool enabling to distinguish spectral components of the observed sequence of values. In some applications such an information is sufficient but in other cases it forms the basis for further processing only. The following chapter presents some methods and algorithms to achieve such an analysis of one or more observed signals based upon the discrete Fourier transform while parametric methods closely connected with random signal theory and signal modelling are discussed later.

4.1 Space-Frequency Analysis

4.1.1 Basic Spectral Estimation

Let us assume at first the problem of the infinite data length analysis. Instead of the discrete Fourier transform discussed before *discrete-time Fourier transform* applied for the infinite sequence $\{x_d(n)\}$ can be used based upon relation [15, p.372]

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_d(n) e^{-j\omega n} \tag{4.1}$$

for continuous frequency ω with its period 2π long. The inverse discrete-time Fourier transform is then defined by relation

$$x_d(n) = \frac{1}{2\pi} \int_0^{2\pi} X_d(\omega) e^{j\omega n} d\omega \tag{4.2}$$

The last expression explains how the given signal is represented by the integral of its harmonic components.

Definition 4.1 *The magnitude and phase spectrum of a sequence* $\{x_d(n)\}_{n=-\infty}^{\infty}$ *is defined as the magnitude and phase of its discrete-time Fourier transform.*

Instead of spectrum sometimes *power spectral density* [22, p.59] is used defined by relation

$$S_{xx}(\omega) = \lim_{L\to\infty} E\left[\frac{1}{2L+1}\left|\sum_{n=-L}^{L} x_d(n) e^{-j\omega n}\right|^2\right] \tag{4.3}$$

with symbol E standing for the mean value of several realizations.

In real cases finite length data segment can be processed only which implies that

- a window function must be used to choose a sequence N samples long
- the discrete Fourier transform can be applied

The main problem of such an approximation is in application of a method which would provide the best estimate of the true function defined for the infinite process realization.

4.1.2 Window Functions

Various window sequences $\{w(n)\}$ of the finite length N are used to derive a finite sequence of the same length from the infinite signal $\{x_d(n)\}$ or impulse response $\{h_d(n)\}$ after their scalar multiplication. Two main applications of windowing include

- the choise of the data segment
$$x(n) = x_d(n)\, w(n) \tag{4.4}$$
to enable the signal analysis
- the evaluation of the finite impulse response filter defined by its impulse response
$$h(n) = h_d(n)\, w(n) \tag{4.5}$$
based on the ideal infinite impulse response.

We shall study now the application of windowing for spectral analysis while its use for filtering will be discussed later.

Using properties of the discrete-time Fourier transform [15, p.379] the scalar multiplication in Eq.(4.4) in time domain is equivalent to the periodic convolution in frequency domain given by relation
$$X(\omega) = \frac{1}{2\pi} X_d(\omega) \otimes W(\omega) \tag{4.6}$$
defined by integral
$$X(\omega) = \frac{1}{2\pi} \int_0^{2\pi} X_d(\Omega) W(\omega - \Omega) d\Omega \tag{4.7}$$

Theorem 4.1 *The discrete-time Fourier transform of sequence*
$$x_d(n) = \sum_i c_i e^{j\omega_i n} \tag{4.8}$$
is given by the linear combination of unit impulses in the form
$$X_d(\omega) = 2\pi \sum_i c_i \delta(\omega - \omega_i) \tag{4.9}$$

Proof: Applying Eq.(4.9) in Eq.(4.2) we obtain
$$x_d(n) = \frac{1}{2\pi} \int_0^{2\pi} 2\pi \sum_i c_i \delta(\omega - \omega_i) e^{j\omega n} d\omega$$
which can be simplified to relation
$$x_d(n) = \sum_i c_i \int_0^{2\pi} \delta(\omega - \omega_i) e^{j\omega n} d\omega$$
providing result in form (4.8).

Example 4.1 *Compare the ideal spectrum of the harmonic sequence* $x_d(n) = \cos(\omega_0 n)$ *and its window approximation.*

Solution: As
$$x_d(n) = \cos(\omega_0 n) = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n}) \tag{4.10}$$
it is possible to apply Theorem 4.1 to evaluate
$$X_d(\omega) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) \tag{4.11}$$

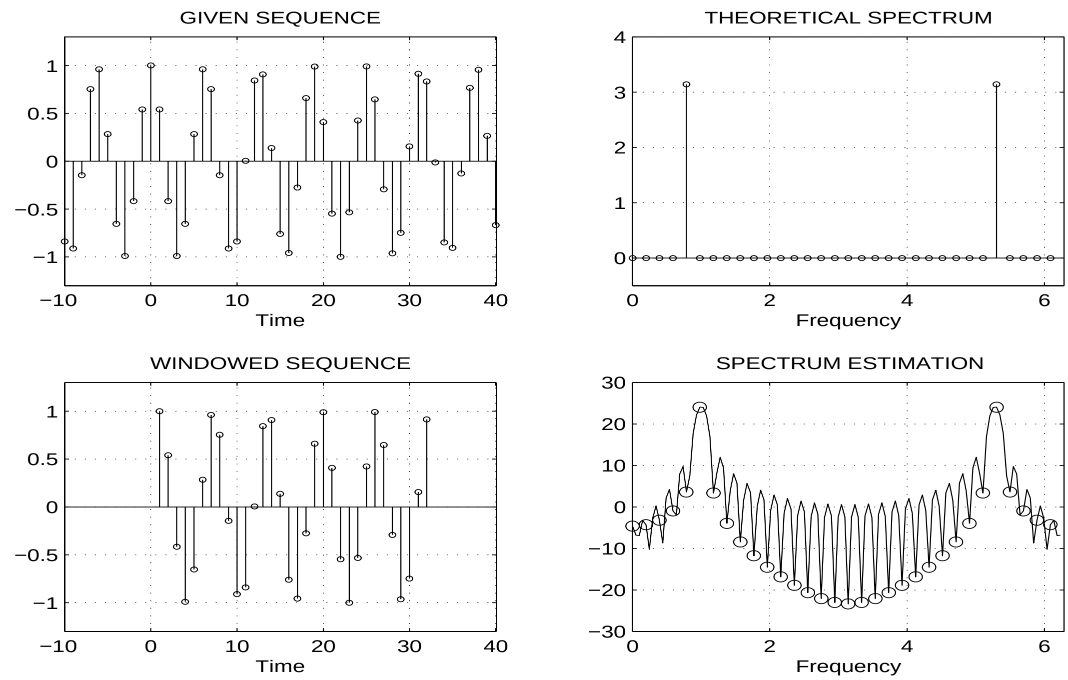

FIGURE 4.1. Theoretical harmonic infinite sequence of frequency $\omega_0 = 1$ with its line spectrum and the comparison with the finite sequence of length $N = 32$ values and its spectrum based upon the discrete-time and discrete Fourier transform.

Using further Eq. (4.6) we can estimate the window spectrum in the form

$$X(\omega) = \frac{1}{2}(W(\omega - \omega_0) + W(\omega + \omega_0)) \tag{4.12}$$

Properties of such a function are in this way entirely dependent upon the window function $\{w(n)\}$. Using the rectangular window defined below it is possible to evaluate the spectrum estimation given in Fig. 4.1.

Definitions of basic window function are summarized in Tab. 4.1 with their sketch and magnitude spectra in Fig. 4.2 based upon Algorithm 4.1. Ideal line spectra of harmonic

Window definition (MATLAB notation)	*Mainlobe width*	*Sidelobe level*
Rectangular (BOXCAR)		
$w(n) = \begin{cases} 1 & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$4\pi/N$	$-13\ dB$
Bartlett (Triangular - BARTLETT)		
$w(n) = \begin{cases} 2n/N - 1 & \text{for } n = 0, 1, \cdots, (N-1)/2 \\ 2 - 2n/(N-1) & \text{for } n = (N-1)/2, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-27\ dB$
Hanning (HANNING)		
$w(n) = \begin{cases} (1 - \cos(2\pi n/(N-1)))/2 & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-32\ dB$
Hamming (HAMMING)		
$w(n) = \begin{cases} 0.54 - 0.46\cos(2\pi n/(N-1)) & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-43\ dB$

TABLE 4.1. Basic window function definition

signal components are according to Eq. (4.12) represented by shifted window spectra (using periodic extension) given in Fig. 4.2. The *mainlobe width* presents the ability to distinguish two closely spaced harmonic components. The *sidelobe level* enables to estimate how small signal can be detected in presence of large ones.

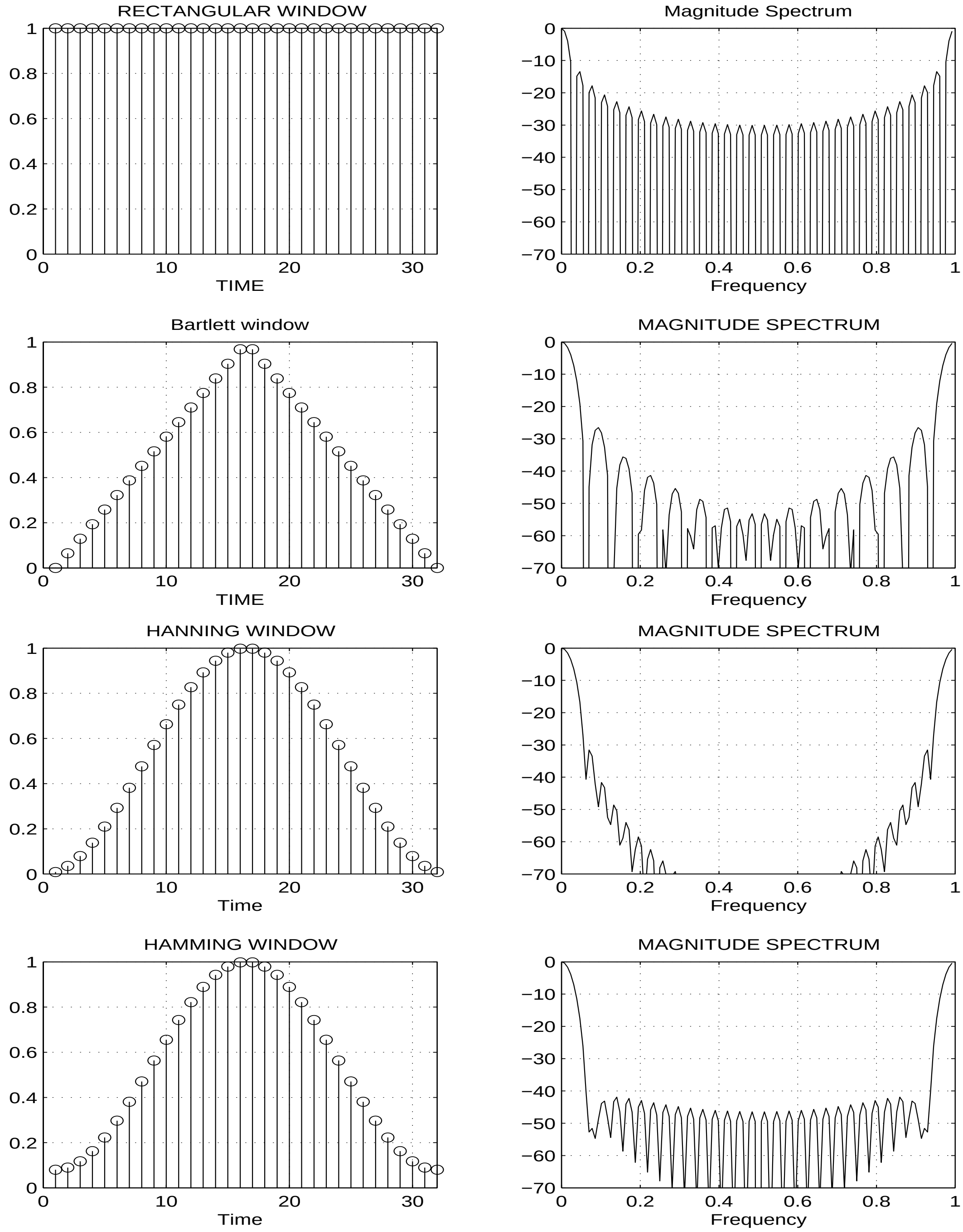

FIGURE 4.2. Basic window functions and their magnitude spectra in dB

Discrete-time Fourier transform enables to evaluate the discrete Fourier transform of a *finite length sequence* by sampling frequency ω over one period in N points. *Spectrum* can be in this way approximated by relation

$$X(k) = X(\omega)\,|_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n} \tag{4.13}$$

Algorithm 4.1 *Basic window functions magnitude spectra evaluation.*

- matrix of window functions definitions for a given length N and their plot

  $\mathbf{w}$ = [boxcar(N) bartlett(N) hanning(N) hamming(N)];
  plot($\mathbf{w}$);

- magnitude spectra evaluation of a chosen length M

  $\mathbf{W}$ =fft($\mathbf{w}, M$);
  $\mathbf{f}$ = $[0 : (M-1)]/M$;
  semilogy($\mathbf{f}$,abs($\mathbf{W}$));

for $k = 0, 1, \cdots, N-1$. The power spectral density function defined by (4.3) can be similarly approximated by *periodogram* defined as

$$S_{xx}(k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-jk\frac{2\pi}{N}n}\right|^2 \tag{4.14}$$

for $k = 0, 1, \cdots, N-1$ taking into account one realization only. Its properties and problems of its application are discussed in [22, p.68].

It is obvious that the spectrum or periodogram estimation based on Eqs. (4.13) and (4.14) can be evaluated using the MATLAB function *fft* mentioned in the previous section.

4.1.3 Short-Time Fourier Transform

In some applications of signal processing with its frequency components changing itself in time it is useful to combine time-domain and frequency domain analysis. In particular it is possible to evaluate the Fourier transform for each time in the neighborhood of that instant [22].

Using the shifted window function $w(m)$ in Eq.(4.4) by n samples and applying Eq. (4.1) we can define the discrete short-time Fourier transform (STFT) by relation

$$X(n,\omega) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)e^{-j\omega m} \tag{4.15}$$

for $n = 0, 1, \cdots$. allowing to choose a short-time section of $\{x(m)\}$ at time n. As N samples long window function implies N samples long time section processing the discrete Fourier transform can be used resulting according to Eq. (4.13) in relation

$$X(n,k) = \sum_{m=n-(N-1)}^{n} X(m)w(n-m)e^{-jk\frac{2\pi}{N}m} \tag{4.16}$$

for any time index n and $k = 0, 1, \cdots, N-1$.

Example 4.2 *Let us apply the short-time Fourier transform to a sequence*

$$x(n) = \sin((\omega_0\, n)n)$$

for $\omega_0 = 0.07$ *and* $n = 0, 1, \cdots, 100$ *using* $N = 32$ *samples long rectangular window function.*

Solution: Applying Eq. (4.16) it is possible to evaluate result given in Fig. 4.3 presenting time dependent signal frequency.

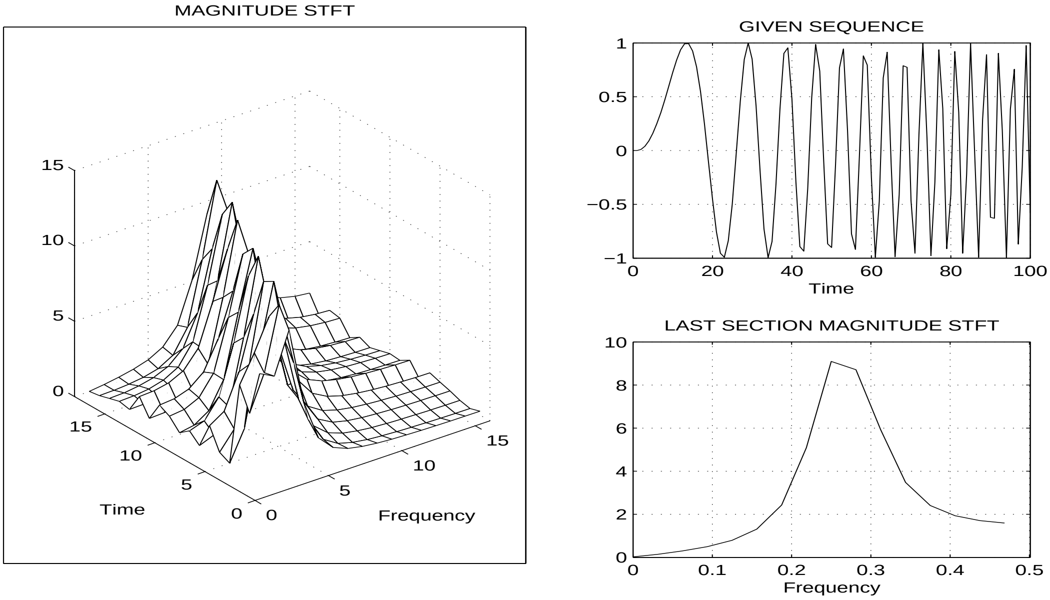

FIGURE 4.3. The short time Fourier transform of sequence $x(n) = \sin((\omega_0\, n)n)$ for $\omega_0 = 0.01$ using rectangular window function $w(n)$ of length $N = 32$ samples and $n = 32, 36, 38, \cdots, 100$.

4.1.4 Cepstral Analysis

Many signal processing methods are based upon presumption that the given system is linear. Such a system is relatively easy to analyse and it is possible to design a variety of useful signal processing functions.

Let us consider now a special class of nonlinear systems with their system transformation H that obey a generalized principle of superposition discussed in more general way in [30] and given by the following relation

$$H[x_1(n)\,(op1)\,x_2(n)] = H[x_1(n)]\,(op1)\,H[x_2(n)] \quad (4.17)$$

$$H[c\,(op2)\,x_1(n)] = c\,(op2)\,H[x_1(n)] \quad (4.18)$$

where $(op1)$ and $(op2)$ stand for any operators satisfying the algebraic postulates of vector addition and scalar multiplication. Such *homomorphic systems* include linear systems as well.

In practical cases it is possible to restrict our interest to systems combining their signals either by multiplication or convolution. Especially the last possibility is studied very often as it can be used for a general problem of echo detection to improve the quality of accoustic signals or analyze behaviour of seismic signals etc.

The main principle of the homomorphic system processing include

- application of the direct characteristic system transforming operator $(op1)$ to addition for signal $\{x(n)\}$ to evaluate $\{c(n)\}$
- processing of signal $\{c(n)\}$ by any linear method
- application of the inverse characteristic system transforming operator of addition back to $(op1)$.

This principle is presented in Fig. 4.4 for systems combining their signals by *convolution*. The *direct characteristic system* is based upon the discrete Fourier transform changing convolution to multiplication, following application of the logarithmic function combining

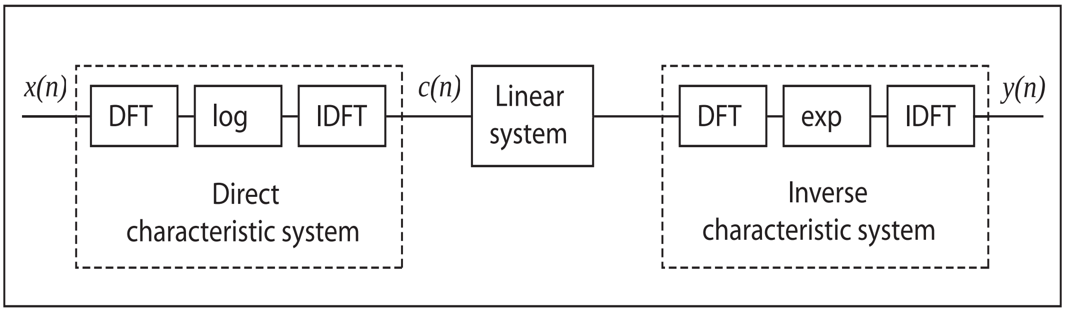

FIGURE 4.4. Representation of a homomorphic system with a complex cepstrum as an output of the direct characteristic system.

signals by addition and inverse Fourier transform returning further processing to the time domain. Resulting signal is called *complex cepstrum* allowing to detect signal echos and their cancelling by a linear system. *Inverse characteristic system* is similar to the direct one with complex exponential instead of logarithm only.

Computer processing of signal $\{x(n)\}$ to obtain its cepstrum is presented in Algorithm 4.2. The whole evaluation can be achieved by the MATLAB function *cceps* as well.

Example 4.3 *Let us analyse cepstrum of signal*

$$x(n) = r(n) \otimes s(n) = \sum_{m=0}^{N-1} r(m)s(n-m)$$

representing given signal and its echo in the form

$$x(n) = r(n) + \alpha r(n - n_1)$$

where

$$\begin{aligned} r(n) &= e^{-0.1n} \sin(\frac{\pi}{4} n) \\ s(n) &= \delta(n) + \alpha \delta(n - n_1) \end{aligned}$$

with $\delta(n)$ standing for an impulse function, $n_1 = 40$ and $\alpha = 0.5$.

Solution: The direct characteristic system involves

- the discrete Fourier transform application

$$X(k) = R(k)S(k)$$

where

$$\begin{aligned} R(k) &= \sum_{n=0}^{N-1} r(n) e^{-jk\frac{2\pi}{N}n} \\ S(k) &= \sum_{n=0}^{N-1} s(n) e^{-jk\frac{2\pi}{N}n} = 1 + \alpha e^{-jk\frac{2\pi}{N}n_1} \end{aligned}$$

- the logarithmic function application

$$\begin{aligned} L(k) &= \log(X(k)) = \log|X(k)| + j \arg(X(k)) = \\ &= \log|R(k)| + \log|S(k)| + j(\arg(R(k) + \arg(S(k))) \end{aligned}$$

The contribution to the complex logarithm due to the impulse train for $|\alpha| < 1$ is

$$L^s(k) = \log\left(1 + \alpha e^{-jk\frac{2\pi}{N}n_1}\right) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i} \left(\alpha e^{-jk\frac{2\pi}{N}n_1}\right)^i$$

Algorithm 4.2 *Cepstral analysis of a given signal* $\{x(n)\}$, $n = 0, 1, \cdots, N-1$ *assuming original signals in convolution.*

- discrete Fourier transform application
  $\mathbf{X}$ =fft($\mathbf{x}$);
- complex logarithm implementation
  $\mathbf{L}$ =log($\mathbf{X}$);
- inverse Fourier transform application
  $\mathbf{c}$ =ifft($\mathbf{L}$);

- the inverse discrete Fourier transform use which for the signal component $s(n)$ provides the contribution to the complex cepstrum in the form

$$c^s(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}\alpha^i e^{-jk(n_1 i - n)\frac{2\pi}{N}} = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{\alpha_i}{i}\delta(n - n_1 i)$$

  As this sequence is additive to the sequence given by the original signal it is obvious that cepstrum can be used for the echo analysis. Results are presented in Fig. 4.5.

Cepstrum component due to the echo can be usually eliminated by a simple window function in the form

$$w(n) = \begin{cases} 0 & \text{for } n < n_i \\ 1 & \text{for } n > n_c \end{cases} \tag{4.19}$$

where n_c stands for the index resulting from the cepstral analysis. Inverse characteristic system can be then used. Results of such a procedure applied to the example given above are presented in Fig. 4.5 as well.

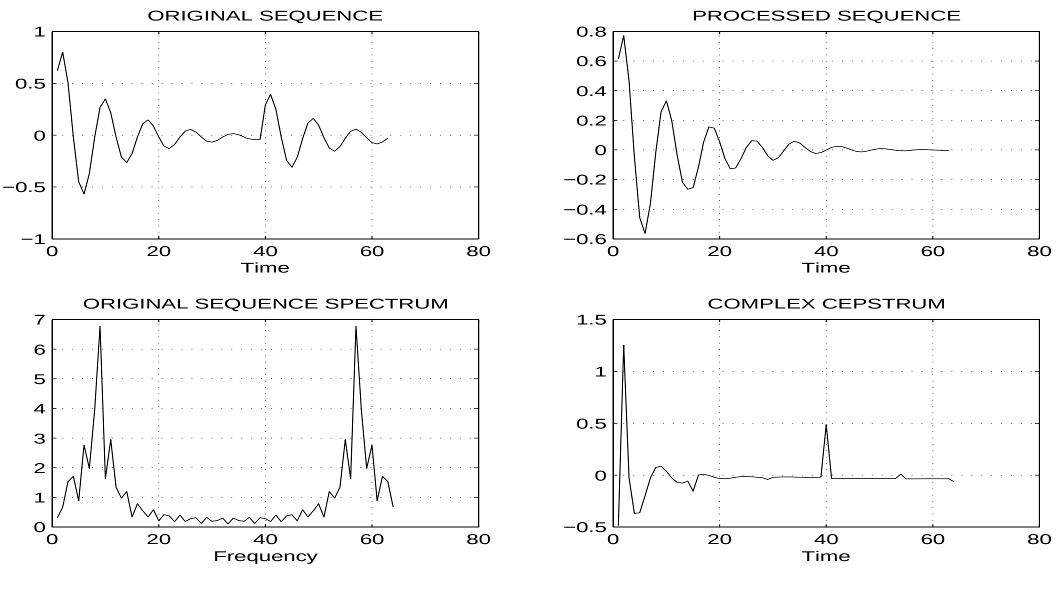

FIGURE 4.5. Original sequence $\{x(n)\}_{n=0}^{N-1}$ composed of sequence $r(n) = e^{-0.1n}\sin(\frac{\pi}{4}n)$ and its echo delayed by $n_1 = 40$ samples attenuated by coefficient $\alpha = 0.5$ for $N = 64$ before and after the homomorphic processing using the window function for its cutoff index $n_c = 30$ and the original sequence magnitude spectrum and complex cepstrum

4.1.5 Two-Dimensional Signal Analysis

While one-dimensional signal analysis and processing can be applied in many engineering and scientific systems producing sequences of values its two-dimensional extension is mainly used in video processing. The mathematical background of such an analysis is based upon discrete space signals $\{x(n_1, n_2)\}$ defined for all integer values of n_1 and n_2. The most important deterministic signals represent

- unit sample sequence: $\delta(n_1, n_2) = \begin{cases} 1 & \text{for } n_1 = n_2 = 0 \\ 0 & \text{otherwise} \end{cases}$
- unit step sequence: $u(n_1, n_2) = \begin{cases} 1 & \text{for } n_1, n_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- real exponential sequence: $x(n_1, n_2) = A\alpha^{n_1}\beta^{n_2}$
- rectangular sequence: $x(n_1, n_2) = \begin{cases} 1 & \text{for } a \leq n_1 \leq b, c \leq n_2 \leq d \\ 0 & \text{otherwise} \end{cases}$

Sketch of signals is given in Fig. 4.6.

It is possible to show [22] that any stable sequence $\{x(n_1, n_2)\}$ can be defined by combination of complex exponentials with coefficients $X(\omega_1, \omega_2)$ according to the following definition.

Definition 4.2 *The direct discrete space Fourier transform is given by relation*

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \tag{4.20}$$

while the inverse discrete space Fourier transform is defined as

$$x(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2 \tag{4.21}$$

Result of the direct *discrete space Fourier transform* is in general a complex function of continuous variables ω_1 and ω_2 having period 2π which implies that

$$X(\omega_1, \omega_2) = X(\omega_1 + 2\pi, \omega_2) = X(\omega_1, \omega_2 + 2\pi) \tag{4.22}$$

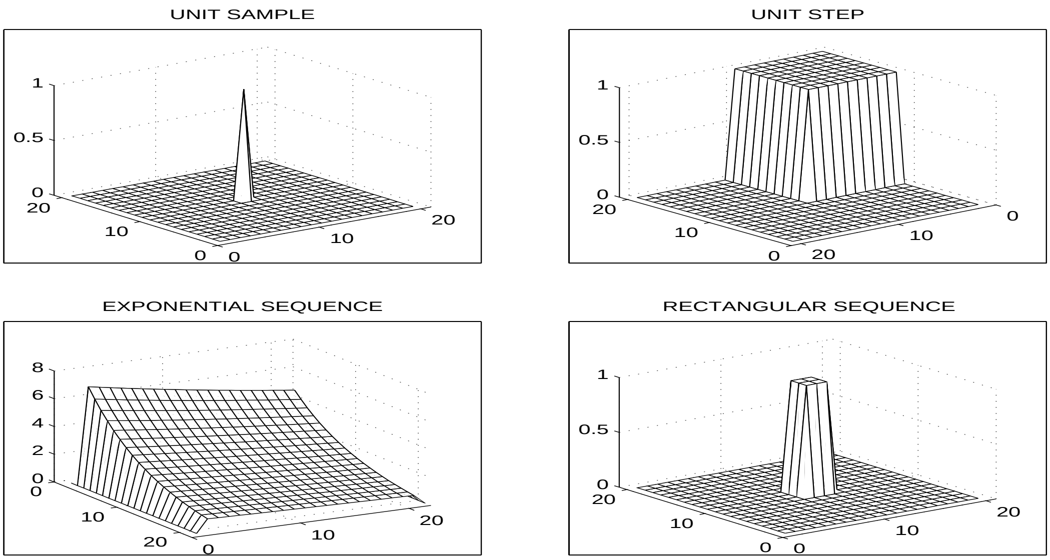

FIGURE 4.6. Basic deterministic two-dimensional signals.

and it represents the signal *spectrum*. In case that $\{x(n_1, n_2)\}$ stands for the two-dimensional linear system impulse response its discrete space Fourier transform represents the *frequency response* of the system.

In many cases the given sequence $\{x(n_1, n_2)\}$ has a finite length with its indices in the range $0 \leq n_1 \leq N_1 - 1$ and $0 \leq n_2 \leq N_2 - 1$. The *discrete Fourier transform* $X(k_1, k_2)$ of such a sequence is related to the discrete space Fourier transform by relation

$$X(k_1, k_2) = X(\omega_1, \omega_2) \,|_{\omega_1 = k_1 \frac{2\pi}{N_1},\, \omega_2 = k_2 \frac{2\pi}{N_2}} \tag{4.23}$$

for $0 \leq k_1 \leq N_1 - 1$, $0 \leq k_2 \leq N_2 - 1$ and it can be evaluated by the following definition.

Definition 4.3 *The direct discrete Fourier transform is given by relation*

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-jk_1 \frac{2\pi}{N_1}} e^{-jk_2 \frac{2\pi}{N_2}} \tag{4.24}$$

for $0 \leq k_1 \leq N_1 - 1$, $0 \leq k_2 \leq N_2 - 1$ *and the inverse discrete Fourier transform is defined as*

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) e^{jk_1 \frac{2\pi}{N_1}} e^{jk_2 \frac{2\pi}{N_2}} \tag{4.25}$$

This definition implies that the sequence $\{x(n_1, n_2)\}$ of length $N_1\, N_2$ in space domain is represented by sequence $\{X(k_1, k_2)\}$ of the same size in frequency domain.

Example 4.4 *Evaluate the discrete-space Fourier transform of the rectangular sequence*

$$x(n_1, n_2) = \begin{cases} 1 & \text{for } -1 \leq n_1 \leq 1, -1 \leq n_2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.26}$$

given in Fig 4.6.

Solution: Using the Definition 4.2 it is possible to find

$$\begin{aligned} X(\omega_1, \omega_2) &= 1 + e^{j\omega_1} e^{-j\omega_2} + e^{-j\omega_2} + e^{-j\omega_1} e^{-j\omega_2} + e^{j\omega_2} + e^{-j\omega_1} + e^{j\omega_1} e^{j\omega_2} + e^{j\omega_2} + e^{-j\omega_1} e^{j\omega_2} = \\ &= 1 + 2(\cos(\omega_1 - \omega_2) + \cos(\omega_1) + \cos(\omega_2) + \cos(\omega_1 + \omega_2)) \end{aligned}$$

Sketch of this (real) function is presented in Fig. 4.7.

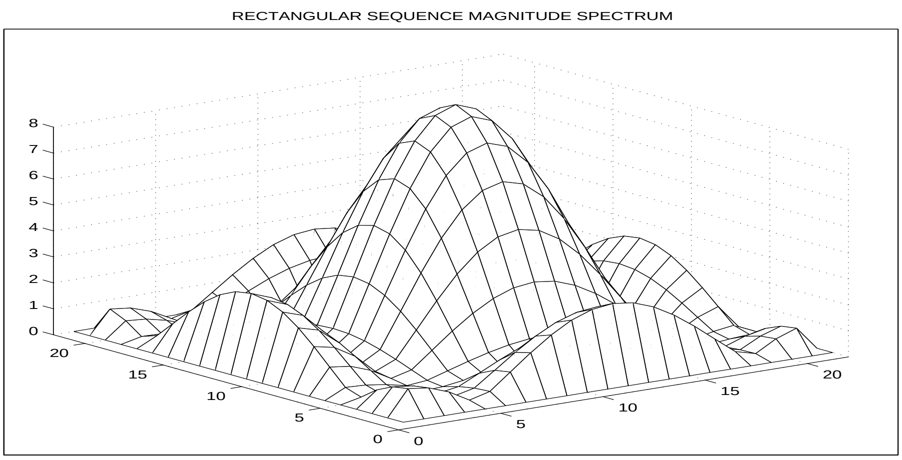

FIGURE 4.7. The discrete space Fourier transform of the rectangular sequence

The application of the two-dimensional Fourier representation of a signal can be used both for signal analysis and its processing using similar principles applied for the one-dimensional case. Further discussion is given in [22].

4.2 Summary

Signal analysis is a very powerful tool to provide spectral estimation of signal components. This chapter presented basic ideas only connected with the discrete Fourier transform and applications of window functions. Further information will be provided in next sections.