Matlab m-files for the Estimation of Non-Gaussian Noise Sample Moments in Wavelet Domain Using the Moment Generating Function

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Abstract

The presented pack of m-files contains several functions for modeling of noise in astronomical and multimedia images. For its favorable properties, we exploit the undecimated wavelet representation. Usually, the noise analysis of the studied imaging system is carried out in the spatial domain. However, noise in astronomical data is non-Gaussian, and thus the noise model parameters need to be estimated directly in the wavelet domain. We derive equations for estimating the sample moments for non-Gaussian noise in the wavelet domain. We consider that the sample moments in the spatial domain are known from the noise analysis and that the model parameters are estimated by using the method of moments.

1 Sample moments in the spatial domain

In the paper [1], the noise model parameters are estimated via the method of moments. Hence, every noise realization n in the spatial domain is described by the sample moments. The rth central sample moment is given by

$$M_r = \frac{1}{I}\sum_{i=1}^{I}(n_i - \overline{n})^r, \quad 1 \leq r \tag{1}$$

where $\overline{n}$ is the mean value. The sample moments of the noise are computed for every acquired video sequence. The choice of moments depends on the used model.

2 Sample moments in the wavelet domain

2.1 Undecimated wavelet transform

For image denoising, the undecimated wavelet transform (UWT) or also the Stationary Wavelet Transform (SWT) is a better choice than the critically sampled discrete wavelet transform

(DWT) [7]. The main reason is that the DWT is shift variant [8] which limits its denoising performance [2]. Wavelet shrinkage methods performed on the DWT coefficients usually cause unwanted artifacts around the objects such as stars [9].

For the reasons explained above, we choose the undecimated wavelet transform [10] for the video frame representation. The UWT is computed using a so-called à trous algorithm [11] which produces the same number of wavelet coefficients at each scale (decomposition level). We use the following notation for the respective wavelet subbands: $\gamma D_{\xi}^{(v)}$, where ξ in the subscript denotes the decomposition level and the superscript in the parentheses denotes the particular detail subband (v-vertical, h-horizontal, or d-diagonal). As we mentioned above, the noise present in astronomical images is non-Gaussian and thus it is necessary to evaluate the sample moments in the wavelet domain. This may be achieved by using the moment generating function. This function of the random variable n (representing the analyzed noise) is closely related to the characteristic function [4] and is defined by

$$M_n(u) = E[e^{un}], \quad u \in \mathbb{R}. \tag{2}$$

The series expansion of e^{un} suggests that the moment generating function allows to find all moments of a given distribution [12]. Provided that the random variable n has a continuous PDF, the $M_n(u)$ is given by

$$\begin{aligned} M_n(u) &= \int_{-\infty}^{\infty} e^{un} p(n) dn \\ &= \int_{-\infty}^{\infty} \left(1 + un + \frac{u^2 n^2}{2!} + \dots\right) p(n) dn \\ &= 1 + u m_1 + \frac{u^2 m_2}{2!} + \dots, \end{aligned} \tag{3}$$

where m_k is the kth moment.

For producing the relations between the moments in the spatial and the wavelet domain, we need to describe the wavelet transform process first. The 1-dimensional (1D) UWT corresponds to convolution filtering of n with the kernel $\boldsymbol{h} = [h_1, h_2 \dots h_k]$ while the down-sampling step is omitted. Hence, each wavelet coefficient is computed as the weighted sum of the independent random variables $n_1, n_2 \dots n_k$ (noise pixels) given as

$$S_k = \sum_{i=1}^{k} h_i n_i. \tag{4}$$

The moment generating function $M_{S_k}(u)$ of S_k then runs as

$$M_{S_k}(u) = M_{n_1}(h_1 u) M_{n_2}(h_2 u) \dots M_{n_k}(h_k u). \tag{5}$$

where

$$M_{n_k}(h_k u) = \left(1 + h_k u m_1 + \frac{(h_k u)^2 m_2}{2!} + \frac{(h_k u)^3 m_3}{3!} + \frac{(h_k u)^4 m_4}{4!} \dots\right). \tag{6}$$

We are going to demonstrate that it is possible to find the sample moments in the wavelet domain by using the values of the sample moments from the spatial domain. As mentioned above, the moment generating function is closely related with the moments of the distribution. Therefore, the rth moment may be evaluated using the moment generating function computed as the rth derivative with respect to the variable u at $u = 0$ given as

$$M_r = M_n^{(r)}(0). \tag{7}$$

Let us consider a zero-mean noise n and its wavelet domain representation $N = \mathcal{UWT}\{n\}$. In our method, we exploit the second and the fourth moment for noise description. These moments are related via the sample kurtosis as demonstrated in [1]. For the sake of simplicity, we assume a short filter such as the Haar filter with the kernel $h = [h_1, h_2]$. The previously stated assumptions suggest the following moment relations. The second sample moment $M_2(N)$ in the wavelet domain is computed from $M_2(n)$ given by

$$M_2(N) = M_2(n) \sum_{i=1}^{2} h_i^2. \tag{8}$$

Similarly the fourth moment $M_4(N)$ computed from $M_4(n)$ given by

$$M_4(N) = 6(M_2(n)h_1h_2)^2 + M_4(n) \sum_{i=1}^{2} h_i^4. \tag{9}$$

Equations (8) and (9) may be generalized for filters with k coefficients $\boldsymbol{h} = [h_1, h_2 \ldots h_k]$ given as

$$M_2(N) = M_2(n) \sum_{i=1}^{k} h_i^2, \tag{10}$$

$$M_4(N) = 6(M_2(n))^2 \left[h_1^2h_2^2 + h_1^2h_3^2 + \ldots + h_1^2h_k^2 + h_2^2h_3^2 + h_2^2h_4^2 + \ldots\right] + M_4(n) \sum_{i=1}^{k} h_i^4. \tag{11}$$

The above equations are demonstrated on the case of the 1D UWT. Nevertheless, the UWT may be easily extended also to the 2-dimensional space. This transform is separable, and thus we carry out the convolution in the row direction and then also in the column direction to obtain the 2D UWT decomposition. The sample moments of the resulting coefficients are evaluated using the derived equations.

2.2 Simplification of the derived equation for the fourth moment

This proposed equation for the fourth moment in the wavelet domain is a little complex, especially for longer wavelet transform filters. We found experimentally that equation (11) could be simplified for certain types of impulsive noise. If we consider the salt and pepper noise for 8-bpp (bits per pixel) images then the probability of the pixel value flipping to 0 is $P(y = 0) = \varepsilon/2$ and the probability of flipping to 255 is $P(y = 255) = \varepsilon/2$. If the parameter ε approximately satisfies $\varepsilon \leq 0.05$ then

$$6(M_2(n))^2 \left[h_1^2h_2^2 + h_1^2h_3^2 + \ldots + h_1^2h_k^2 + h_2^2h_3^2 + h_2^2h_4^2 + \ldots\right] \ll M_4(n) \sum_{i=1}^{k} h_i^4. \tag{12}$$

As a result, equation (11) can be simplified as

$$M_4(N) = M_4(n) \sum_{i=1}^{k} h_i^4. \qquad (13)$$

We tested these findings also on the 16-bpp dark frames [13] which were acquired by the SBIG ST-8 astronomical camera as described in [1]. Our experiments indicate that these dark frames may be modelled as white impulsive noise for which the hot pixels do not always reach the maximum value of the dynamic range. Using this model, (12) is satisfied for all the acquired dark frames within the whole temperature range (from 268.15 to 293.15 K).

3 Implementation in Matlab programming environment

Proposed equations (PROPOSED METHOD) were implemented in Matlab. The moment-generating function given by equations (5) and (6) is implemented in the Matlab function *MGFun.m*. This function can be modified in accordance with required length of the filter and required moment order. Matlab function *M4Fun.m* contains the equation (9) and (11) for the evaluation of the fourth sample moment of an independent random variable convolved with kernel.

Demonstration of the proposed algorithm can be found in m-file *demoMGF.m*. This m-file demonstrates the utilization of *M4Fun.m* in the process of estimation of fourth sample moment in the undecimated wavelet domain. User can choose the real data represented by dark frame acquired by astronomical camera (exposure time 60 s, temperature 293.15 K) and artificial data represented by uniformly distributed pseudo-random numbers.

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References

[1] Svihlik, J., Fliegel, K., Kukal, J., Jerhotova, E., Pata, P., Vitek, S., and Koten, P., Estimation of Non-Gaussian noise parameters in the wavelet domain using moment generating function, *J. Electron. Imaging* **21**(1) (2012).

[2] A. Pizurica, "Image denoising using wavelets and spatial context modeling," Ph.D. dissertation, Univ. Gent, Gent, Belgium, (2002).

[3] J. Švihlík and K. Fliegel and P. Koten and S. Vítek and Páta, "Noise Analysis of MAIA System and Possible Noise Suppression," *Radioengineering* **20**(1), 110-117 (2011).

[4] W. B. Davenport, *Probability and Random Processes: An Introduction for Applied Scientists and Engineers*, New York, Prentice Hall, Inc., (1970).

[5] Ch. Buil, *CCD Astronomy*, Richmond, Willman-Bell, Inc., (1991).

[6] http://www.sbig.com/

[7] M. Raphan and E. P. Simoncelli, "Optimal denoising in redundant representations," *IEEE Transactions on Image Processing* **17**(8) 1342-1352, (2008).

[8] S. G. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. on Pattern Anal. and Mach. Intell.* **2**(7), 674-693 (1989).

[9] J. L. Starck and F. Murtagh and A. Bijaoui, *Image processing and data analysis, The multiscale approach*, Cambridge University Press, (1998).

[10] http://www-ece.rice.edu/dsp/software/RWT

[11] M. Holschneider and J. Kronland-Martinet and R. J. Morlet and P. Tchamitchian, *Wavelets, time-frequency methods and phase space, Real Time Algorithm for Signal Analysis with the Help of the Wavelet Transform.* Springer-Verlag, Berlin, (1989).

[12] M. G. Bulmer, *Principles of Statistics*, Mineola, Dover Publications, Inc., (1979).

[13] J. Švihlík, "Modeling of Scientific Images Using GMM," *Radioengineering* **18**(4), 579-586 (2009).